Possible experimental test of
continuous medium model for fractal media

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Abstract

We use the fractional integrals to describe fractal media. We consider the fractal media as special ("fractional") continuous media. We discuss the possible experimental testing of the continuous medium model for fractal media that is suggested in [Phys. Lett. A 336 (2005) 167]. This test is connected with measure of period of the Maxwell pendulum with fractal medium cylinder.
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1. Introduction

The real structures of the media are characterized by an extremely complex and irregular geometry [1]. Because the methods of Euclidean geometry, which ordinarily deals with regular sets, are purely suited for describing objects such as in nature, stochastic models are taken into account [2]. Another possible way of describing a complex structure of the media is to use fractal theory of sets of fractional dimensionality [1]. Although the fractal dimensionality does not reflect completely the geometric properties of the fractal, it nevertheless permits a number of important conclu-

sions about the behavior of fractal structures [3,4]. For example, if it is assumed that matter with a constant density is distributed over the fractal, then the mass of the fractal enclosed in a volume of characteristic dimension R satisfies the scaling law $M(R) \sim R^D$, whereas for a regular n-dimensional Euclidean object $M(R) \sim R^n$. Let us assume that a medium can be treated on a scale R as a stochastic fractal of dimensionality $D < 3$ embedded in a Euclidean space of dimensionality $n = 3$. Naturally, in real objects the fractal structure cannot be observed on all scales. For example, Katz and Thompson [5] presented experimental evidence indicating that the pore spaces of a set of sandstone samples are fractals in length extending from 10 angstrom to 100 μm.

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In the general case, the fractal media cannot be considered as continuous media. There are points and domains that are not filled of the medium particles. We suggest to consider the fractal media as special (fractional) continuous media. We use the procedure of replacement of the fractal medium with fractal mass dimension by some continuous medium that is described by fractional integrals. This procedure is a fractional generalization of Christensen approach [6]. Suggested procedure leads to the fractional integration and differentiation to describe fractal media. The fractional integrals allow us to take into account the fractality of the media. In order to describe the fractal media by continuous medium model, we must use the fractional integrals. The order of fractional integral is equal to the fractal mass dimension of the media. More consistent approach to describe the fractal media is connected with the mathematical definition the integrals on fractals. In Ref. [8] was proved that integrals on net of fractals can be approximated by fractional integrals. Therefore, we can consider the fractional integrals as approximation of the integrals on fractals [7]. In Ref. [9], we proved that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor. To prove this statement, we use the well-known formulas of dimensional regularizations [10].

We use the fractional integrals to describe fractal media. We consider the simple example of application of fractal integrals to describe the fractal medium. We discuss the possible experimental testing of the continuous medium model for fractal media that is suggested in [12]. This test is connected with measure of period of the Maxwell pendulum with fractal medium cylinder.

It is known that the homogeneous medium cylinder with mass M and the radius R such that the medium of cylinder has the integer mass dimension, has the moments of inertia

$$I_z^{(2)} = (1/2)MR^2,$$

where z is a cylinder axis. If the fractal medium cylinder has the mass M, radius R, and the fractal mass dimension, then the moment of inertia has the form

$$I_z^{(\alpha)} = \frac{\alpha}{\alpha + 2}MR^2, \tag{1}$$

where α is a fractal mass dimension of cross-section of the cylinder ($1 < \alpha \leqslant 2$). The parameter α can be calculated by box counting method for the cross-section of the cylinder.

As the result, we get that two homogeneous cylinders with the same masses and radiuses have the unequal moments of inertia. These cylinders have the different properties as Maxwell pendulums. The periods of oscillation for these pendulums are connected by the equation

$$\left(\frac{T_0^{(\alpha)}}{T_0^{(2)}}\right)^2 = \frac{4(\alpha + 1)}{3(\alpha + 2)},$$

where $T_0^{(\alpha)}$ is a period of oscillation of fractal medium Maxwell pendulum. For example, the experiment can be realized by using the sandstone cylinder. Note that Katz and Thompson [5] presented experimental evidence indicating that the pore spaces of a set of sandstone samples are fractals and are self-similar over three to four orders of magnitude in length extending from 10 angstrom to 100 μm.

2. Fractal media and fractional integrals

The cornerstone of fractals is the meaning of dimension, specifically the fractal dimension. Fractal dimension can be best calculated by box counting method which means drawing a box of size R and counting the mass inside. The mass fractal dimension [1,11] can be easy measured for fractal media. The properties of the fractal media like mass obeys a power law relation

$$M \sim R^D \quad (D < 3), \tag{2}$$

where M is the mass of fractal medium, R is a box size (or a sphere radius), and D is a mass fractal dimension. Amount of mass of a medium inside a box of size R has a power law relation (2).

The power law relation (2) can be naturally derived by using the fractional integral. In Ref. [12], we prove that the mass fractal dimension is connected with the order of fractional integrals. Therefore the fractional integrals can be used to describe fractal media with non-integer mass dimensions.

Let us consider the region W_A in 3-dimensional Euclidean space E^3, where A is the midpoint of this

region. The volume of the region W_A is denoted by $V(W_A)$. If the region W_A is a ball with the radius R_A, then the midpoint A is a center of the ball, and the volume $V(W_A) = (4/3)\pi R_A^3$. The mass of the region W_A in the fractal medium is denoted by $M(W_A)$. The fractality of medium means that the mass of this medium in any region W_A of Euclidean space E^3 increase more slowly that the volume of this region. For the ball region of the fractal medium, this property can be described by the power law (2), where R is the radius of the ball W_A.

Fractal media are called homogeneous fractal media if the power law (2) does not depend on the translation of the region. The homogeneity property of the medium can be formulated in the form: For all regions W_A and W_B of the homogeneous fractal medium such that the volumes are equal $V(W_A) = V(W_B)$, we have that the masses of these regions are equal $M(W_A) = M(W_B)$. Note that the wide class of the fractal media satisfies the homogeneous property. In many cases, we can consider the porous media [14,15], polymers [16], colloid aggregates [17], and aerogels [18] as homogeneous fractal media.

In Ref. [12], the continuous medium model for the fractal media was suggested. Note that the fractality and homogeneity properties can be realized in the following forms:

(1) Fractality: The mass of the ball region W of fractal medium obeys a power law relation

$$M_D(W) = M_0\left(\frac{R}{R_c}\right)^D, \qquad (3)$$

where $D < 3$, R is the radius of the ball, and R_c is the characteristic value. In the general case, we have the scaling law relation $M_D(\lambda W) = \lambda^D M_D(W)$, where $\lambda W = \{\lambda \mathbf{x}, \ \mathbf{x} \in W\}$.

(2) Homogeneity: The local density of homogeneous fractal media is translation invariant value that have the form $\rho(\mathbf{x}) = \rho_0 = \text{const}$.

These requirements can be realized by the fractional generalization of the equation

$$M_3(W) = \int_W \rho(\mathbf{x}) \, dV_3. \qquad (4)$$

Let us define the fractional integral in Euclidean space E^3 in the Riesz form [19]. The fractional generalization of Eq. (4) can be realized in the following form

$$M_D(W) = \int_W \rho(\mathbf{x}) \, dV_D, \qquad (5)$$

where $dV_D = c_3(D, \mathbf{x}) \, dV_3$, and

$$c_3(D, \mathbf{x}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{x}|^{D-3}.$$

Here we use the initial points in the fractional integrals are set to zero. The numerical factor in Eq. (5) has this form in order to derive usual integral in the limit $D \to (3-0)$. Note that the usual numerical factor $\gamma_3^{-1}(D) = \Gamma(1/2)/2^D \pi^{3/2}\Gamma(D/2)$, which is used in Ref. [19] leads to $\gamma_3^{-1}(3-0) = \Gamma(1/2)/2^3 \pi^{3/2}\Gamma(3/2)$ in the limit $D \to (3-0)$.

In order to have the usual dimensions of the physical values, we can use vector $\mathbf{x}$, and coordinates x, y, z as dimensionless values. For example, we can define $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$, where x_1, x_2, and x_3 are the coordinates with the usual physical dimension.

We can rewrite Eq. (5) in the form

$$M_D(W) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W \rho(\mathbf{x})|\mathbf{x}|^{D-3} \, d^3x. \qquad (6)$$

If we consider the homogeneous fractal medium $(\rho(\mathbf{x}) = \rho_0 = \text{const})$ and the ball region $W = \{\mathbf{x}: |\mathbf{x}| \leqslant R\}$, then we have

$$M_D(W) = \rho_0 \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W |\mathbf{x}|^{D-3} \, dV_3.$$

Using the spherical coordinates, we get

$$\begin{aligned} M_D(W) &= \frac{\pi 2^{5-D}\Gamma(3/2)}{\Gamma(D/2)} \rho_0 \int_W |\mathbf{x}|^{D-1} \, d|\mathbf{x}| \\ &= \frac{2^{5-D}\pi\Gamma(3/2)}{D\Gamma(D/2)} \rho_0 R^D. \end{aligned}$$

As the result, we have $M(W) \sim R^D$, i.e., we derive Eq. (3) up to the numerical factor. Therefore the fractal medium with non-integer mass dimension D can be described by fractional integral of order D.

Note that the interpretation of the fractional integration is connected with fractional dimension [9].

This interpretation follows from the well-known formulas for dimensional regularizations [10]: The fractional integral can be considered as a integral in the fractional dimension space up to the numerical factor $\Gamma(D/2)/(2\pi^{D/2}\Gamma(D))$.

3. Moment of inertia of fractal medium cylinder

In this section, we consider the fractional generalization of the equation for moment of inertia.

The equation for the moment of inertia of homogeneous cylinder with integer mass dimension has the well-known form

$$I_z^{(2)} = \rho_0 \int_S \left(x^2 + y^2\right) dS_2 \int_L dz. \tag{7}$$

Here z is the cylinder axis, and $dS_2 = dx\,dy$. The fractional generalization of Eq. (7) for moment of inertia can be defined by the equation

$$I_z^{(\alpha)} = \rho_0 \int_S \left(x^2 + y^2\right) dS_\alpha \int_L dl_\beta, \tag{8}$$

where we use the following notations

$$dS_\alpha = c(\alpha)\left(\sqrt{x^2 + y^2}\right)^{\alpha-2} dS_2,$$

$$dS_2 = dx\,dy, \qquad c(\alpha) = \frac{2^{2-\alpha}}{\Gamma(\alpha/2)},$$

$$dl_\beta = \frac{|z|^{\beta-1}}{\Gamma(\beta)} dz. \tag{9}$$

The numerical factor in Eq. (8) has this form in order to derive usual integral in the limit $\alpha \to (2-0)$ and $\beta \to (1-0)$. The parameters α and β are

$$1 < \alpha \leqslant 2, \qquad 0 < \beta \leqslant 1.$$

If $\alpha = 2$ and $\beta = 1$, then Eq. (8) has form (7). The parameter α is a fractal mass dimension of the cross-section of cylinder. This parameter can be easy calculated from the experimental data. It can be calculated by box counting method for the cross-section of the cylinder.

Substituting Eq. (9) in Eq. (8), we get

$$I_z^{(\alpha)} = \frac{\rho_0 c(\alpha)}{\Gamma(\beta)} \int_S \left(x^2 + y^2\right)^{\alpha/2} dS_2 \int_0^H z^{\beta-1} dz.$$

Here we consider the cylindrical region W that is defined by the relations

$$L = \{z:\, 0 \leqslant z \leqslant H\},$$
$$S = \left\{(x,y):\, 0 \leqslant x^2 + y^2 \leqslant R^2\right\}. \tag{10}$$

Using the cylindrical coordinates (ϕ, r, z), we have

$$dS_2 = dx\,dy = r\,dr\,d\phi, \qquad \left(x^2 + y^2\right)^{\alpha/2} = r^\alpha.$$

Therefore the moment of inertia is defined by

$$I_z^{(\alpha)} = \frac{2\pi\rho_0 c(\alpha)}{\Gamma(\beta)} \int_0^R r^{\alpha+1}\, dr \int_0^H z^{\beta-1}\, dz$$

$$= \frac{2\pi\rho_0 c(\alpha)}{(\alpha+2)\beta\Gamma(\beta)} R^{\alpha+2} H^\beta.$$

As the result, we have

$$I_z^{(\alpha)} = \frac{2\pi\rho_0 c(\alpha)}{(\alpha+2)\beta\Gamma(\beta)} R^{\alpha+2} H^\beta. \tag{11}$$

If $\alpha = 2$ and $\beta = 1$, we get $I_z^{(2)} = (1/2)\pi\rho_0 R^4 H$.

The mass of the usual medium cylinder (10) is defined by the equation

$$M = \rho_0 \int_S dS_2 \int_L dz$$

$$= 2\pi\rho_0 \int_0^R r\,dr \int_0^H dz = \pi\rho_0 R^2 H. \tag{12}$$

We can consider the fractional generalization of this equation. The mass of fractal medium cylinder (10) can be defined by the equation

$$M_\alpha = \rho_0 \int_S dS_\alpha \int_L dl_\beta. \tag{13}$$

Using the cylindrical coordinates, we get

$$M_\alpha = \frac{2\pi\rho_0 c(\alpha)}{\Gamma(\beta)} \int_0^R r^{\alpha-1}\, dr \int_0^H z^{\beta-1}\, dz$$

$$= \frac{2\pi\rho_0 c(\alpha)}{\alpha\beta\Gamma(\beta)} R^\alpha H^\beta.$$

As the result, we have

$$M_\alpha = \frac{2\pi\rho_0 c(\alpha)}{\alpha\beta\Gamma(\beta)} R^\alpha H^\beta. \tag{14}$$

Substituting the mass (14) in the moment of inertia (11), we get the relation

$$I_z^{(\alpha)} = \frac{\alpha}{\alpha + 2} M_\alpha R^2. \tag{15}$$

Note that Eq. (15) does not have the parameter β. If $\alpha = 2$, we have the well-known relation $I_z^{(2)} = (1/2)MR^2$ for the homogeneous cylinder that has the integer mass dimension $D = 3$ and $\alpha = 2$.

If we consider the fractal medium cylinder with the mass and radius that are equal to mass and radius of the homogeneous medium cylinder with integer mass dimension, then the moments of inertia of these cylinders are connected by the equation

$$I_z^{(\alpha)} = \frac{2\alpha}{\alpha + 2} I_z^{(2)}. \tag{16}$$

Here $I_z^{(2)}$ is the moment of inertia for the cylinder with integer mass dimension $D = 3$ and $\alpha = 2$. For example, the parameter $\alpha = 1.5$ leads us to the relation $I^{(3/2)} = (6/7)I_z^{(2)}$. Using $1 \leqslant \alpha \leqslant 2$, we have the general relation

$$(2/3) \leqslant I_z^{(\alpha)}/I_z^{(2)} \leqslant 1.$$

4. Equation of motion for Maxwell pendulum

Maxwell pendulum is used to demonstrate transformations between gravitational potential energy and rotational kinetic energy. Wind the string up on the small axis, giving the device some initial gravitational potential energy. When released, this gravitational potential energy is converted into rotational kinetic energy, with a lesser amount of translational kinetic energy. We consider the Maxwell pendulum as a cylinder that is suspended by string. The string is wound on the cylinder.

The equations of motion for Maxwell pendulum have the form

$$M_\alpha \frac{dv_y}{dt} = M_\alpha g - T, \qquad I_z^{(\alpha)} \frac{d\omega_z}{dt} = RT,$$

where g is the acceleration such that $g \simeq 9.81$ (m/s^2); the axis z is a cylinder axis, T is a string tension. Using $v_y = \omega_z R$, we have

$$M \frac{dv_y}{dt} = Mg - \frac{I_z^{(\alpha)}}{R^2} \frac{dv_y}{dt}.$$

As the result, we get the acceleration of the cylinder

$$a_y = \frac{dv_y}{dt} = \frac{Mg}{M + I_z^{(\alpha)}/R^2}$$
$$= \frac{g}{1 + I_z^{(\alpha)}/(MR^2)}. \tag{17}$$

Substituting Eq. (15) in Eq. (17), we get

$$a_y = \frac{\alpha + 2}{2\alpha + 2} g. \tag{18}$$

For the fractal mass dimension of cross-section of cylinder $\alpha = 1.5$, we get $a_y = (3/5)g \simeq 6.87$ (m/s^2). For the cylinder with integer mass dimension of the cross-section ($\alpha = 2$), we have $a_y = (2/3)g \simeq 6.54$ (m/s^2). The period T_0 of oscillation for this Maxwell pendulum is defined by the equation

$$T_0 = 4t_0 = 4\sqrt{2L/a_y},$$

where L is a string length, and the time t_0 satisfies the equation $a_y t_0^2/2 = L$. Therefore, we get the relation for the periods

$$\frac{T_0^{(\alpha)}}{T_0^{(2)}} = \sqrt{a_y^{(2)}/a_y^{(\alpha)}} = \sqrt{\frac{4(\alpha + 1)}{3(\alpha + 2)}}. \tag{19}$$

Using $1 < \alpha < 2$, we can see that

$$(8/9) < \left(T_0^{(\alpha)}/T_0^{(2)}\right)^2 < 1.$$

Note the parameter α can be calculated by box counting method for the cross-section of the cylinder. For $\alpha = 1.5$, we have $(T_0^{(\alpha)}/T_0^{(2)})^2 = 0.952$.

5. Conclusion

In the general case, the fractal media cannot be considered as continuous media. There are points and domains that are not filled of the medium particles. In Ref. [12], we suggest to consider the fractal media as special ("fractional") continuous media. We use the procedure of replacement of the medium with fractal mass dimension by some continuous medium that is described by fractional integrals. This procedure is a fractional generalization of Christensen approach [6]. Suggested procedure leads to the fractional integration and differentiation to describe fractal media. The fractional integrals are considered as approximation of the integrals on fractals [8]. Note that fractional integrals

can be considered as integrals over the space with fractional dimension up to numerical factor [9,20,22]. To prove we use the well-known formulas of dimensional regularizations [10]. The fractional integrals allow us to take into account the fractality of the media.

The fractional continuous models of fractal media can have a wide application. This is due in part to the relatively small numbers of parameters that define a random fractal medium of great complexity and rich structure. In many cases, the real fractal structure of matter can be disregarded and the medium can be replaced by some "fractional" continuous mathematical model. In order to describe the media with non-integer mass dimension, we must use the fractional calculus. Smoothing of the microscopic characteristics over the physically infinitesimal volume transform the initial fractal medium into "fractional" continuous model that uses the fractional integrals. The order of fractional integral is equal to the fractal mass dimension of the medium. The fractional continuous model allows us to describe dynamics for wide class fractal media [13,21,23].

In this Letter we consider the simple experiment that allows us to prove the fractional continuous media model [12] for fractal media. This simple experiment can prove that the fractional integrals can be used to describe fractal media.

The fractal medium cylinder with the mass M, radius R, and the fractal mass dimension, has the moment of inertia in the form

$$I_z^{(\alpha)} = \frac{\alpha}{\alpha + 2} M R^2,$$

where α is a fractal mass dimension of cross-section of the cylinder ($1 < \alpha \leqslant 2$). The parameter α can be calculated by box counting method for the cross-section of the cylinder. As the result, we get that two homogeneous cylinders with equal masses and radiuses can have the unequal moments of inertia. These cylinders have the different properties as Maxwell pendulums. The periods of oscillation for these pendulums are connected by the equation

$$\left(\frac{T_0^{(\alpha)}}{T_0^{(2)}} \right)^2 = \frac{4(\alpha + 1)}{3(\alpha + 2)},$$

where $T_0^{(\alpha)}$ is a period of oscillation of fractal medium Maxwell pendulum. For example, the experiment can

be realized by using the sandstone. Note that Katz and Thompson [5] presented experimental evidence indicating that the pore spaces of a set of sandstone samples are fractals and are self-similar over three to four orders of magnitude in length extending from 10 angstrom to 100 μm. The deviation $T_0^{(\alpha)}$ from $T_0^{(2)}$ is no more that 6 per cent. Therefore the precision of the experiments must be high.

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