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Fractal electrodynamics via non-integer dimensional space approach

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ABSTRACT

Using the recently suggested vector calculus for non-integer dimensional space, we consider electrodynamics problems in isotropic case. This calculus allows us to describe fractal media in the framework of continuum models with non-integer dimensional space. We consider electric and magnetic fields of fractal media with charges and currents in the framework of continuum models with non-integer dimensional spaces. An application of the fractal Gauss's law, the fractal Ampere's circuital law, the fractal Poisson equation for electric potential, and equation for fractal stream of charges are suggested. Lorentz invariance and speed of light in fractal electrodynamics are discussed. An expression for effective refractive index of non-integer dimensional space is suggested.

1. Introduction

Fractal electrodynamics based on continuum models of fractal distribution of charges, currents and fields has been suggested in [1–5] ten years ago. These continuum models use the concept of power-law density of states and an application of fractional-order integration. It has been proved that D-order integration is connected with D-dimensional integration [4]. Then these continuum models of fractal electrodynamics have been applied and developed in two directions: (a) fractional integral models by Baskin and Iomin [6,7], by Ostoja-Starzewski [8] to describe anisotropic fractal cases; (b) fractional (non-integer) dimensional models by Muslih, Baleanu and coauthors [9–11], by Zubair, Mughal, Naqvi [12–16], by Balankin with coauthors [17], to describe an anisotropic case, multipoles, and electromagnetic waves in fractional space. Effective continuum models of fractal electrodynamics, which is considered in papers [9–17], are based on Stillinger and Palmer–Stavrinou generalizations of the scalar Laplacian that are suggested in [18] and [19], respectively. In these papers [18,19], the authors have proposed only the second order differential operators for scalar fields in the form of the scalar Laplacian in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian are not considered in [18,19].

Possibility to use only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models of fractal media. For example, Stillinger's form of Laplacian cannot be used for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic fields $\mathbf{B}(\mathbf{r},t)$ in electrodynamic continuum models with non-integer dimensional spaces.

In recent paper [21], it was suggested a generalization of vector differential operators of first orders (gradient, divergence, curl operators) and the vector Laplacian for non-integer dimension spaces. This allows us to extend the scope of possible applications of continuum models with non-integer dimensional spaces. Using this new tool we can describe isotropic fractal media by using the non-integer dimensional space approach.

For anisotropic fractal case, an attempt to suggest D-dimensional vector operations of first order has been presented in the works [12–17]. In these papers, the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Recently [22] a generalization of gradient, divergence, and curl has been suggested without any approximation. The strict approach to continuum models of anisotropic fractal media by the vector calculus on non-integer dimensional space has been described in [22], where a review of different approaches are also suggested.

It should not be confused fractal electrodynamics and fractional electrodynamics that is based on fractional-order vector calculus [23]. Note that first time the fractional calculus has been applied in the electrodynamics by Joseph Liouville about two hundred years ago [24]. An attempt to use a fractional calculus in electrodynamics by introducing some differential vector operations was made by Engheta [25–28]. In these papers, the fractional integral vector operations and fractional generalization of integral theorems of Green, Stokes and Gauss are not considered. A rigorous self-consistent formulation of fractional differential and integral vector calculus was suggested in [23]. The fractional-order differential and integral vector operations are mutually agreed by the

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use of the Caputo fractional derivatives as an inverse operation to the Riemann–Liouville fractional integration. Using this inter-consistency of fractional differential and integral vector operators, the fractional Green's, Stokes' and Gauss's theorems have been are proved. We can note that the theory of fractional-order derivatives and integrals has be applied to several specific electromagnetic problems (for example, see [29–37]).

It should be also noted that the term "fractal electrodynamics" is used in the narrow sense in engineering [38–40] to describe fractal antennas, arrays and apertures and electromagnetic wave scattering from fractal surfaces. We use this term in a broader sense to the theory of fractal distribution of charges, currents, fields, and to electrodynamics of fractal media, and electromagnetic fields on fractal sets.

In this paper, we demonstrate an application of the vector calculus on non-integer dimensional space, which is suggested in [21], to fractal electrodynamics in isotropic case. We give an application of the fractal Gauss's law, the fractal Ampere's circuital law, the fractal Poisson equation for electric potential, and equation for fractal stream of charges.

2. *D*-dimensional integration and differentiation

Let us give some introduction to noninteger-dimensional integration and differentiation of integer orders (for details, see [18–22]).

The D-dimensional integration (see [18] and Section 4 of [20]) for scalar functions $f(\mathbf{r}) = f(|\mathbf{r}|)$ can be defined in terms of ordinary integration by the expression

$$\int d^D\mathbf{r}\, f(\mathbf{r}) = \int\limits_{\Omega_{D-1}} d\Omega_{D-1} \int\limits_0^\infty dr\, r^{D-1}\, f(r), \tag{1}$$

where we can use

$$\int\limits_{\Omega_{D-1}} d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = S_{D-1}. \tag{2}$$

For integer $D = n$, equation (2) gives the well-known area S_{n-1} of $(n-1)$-sphere with unit radius.

As a result, the explicit expressions [20] of D-dimensional integration for arbitrary non-integer D has the form

$$\int d^D\mathbf{r}\, f(|\mathbf{r}|) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int\limits_0^\infty dr\, r^{D-1}\, f(r). \tag{3}$$

This equation reduced D-dimensional integration to ordinary one-dimensional integration. It is obvious that the linearity and translation invariance follow from linearity and translation invariance of ordinary integration. The scaling and rotation covariance can also be derived from equation (3).

In the continuum models of fractal media, it is convenient to work with the physically dimensionless variables $x/R_0 \to x$, $y/R_0 \to x$, $z/R_0 \to x$, $\mathbf{r}/R_0 \to \mathbf{r}$, that yields dimensionless integration and dimensionless differentiation in D-dimensional space. In this case the physical quantities of fractal media have correct physical dimensions.

The volume of D-dimensional ball V_D of radius R Is given by the expression

$$|V_D| = \frac{\pi^{D/2}}{\Gamma(D/2+1)} R^D, \tag{4}$$

and surface area of the d-dimensional sphere S_d of radius R is given by

$$|S_d| = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)} R^d. \tag{5}$$

In general, the dimension d of the boundary $S_d = \partial V_D$ of the region V_D of fractal medium and the dimension D of the region V_D are not related by the equation $d = D-1$. The difference between D and d defines a radial dimension $\alpha_r = D - d$ of the fractal medium. If the radial dimension is equal to one, then (5) can be represented by

$$|S_d| = \frac{2\pi^{D/2}}{\Gamma(D/2)} R^{D-1}. \tag{6}$$

The vector differential operators for non-integer dimension have been derived in [21] by analytic continuation in dimension from integer n to non-integer D.

For simplification we will consider two following cases:

1) *Spherically symmetric case of fractal media*, where scalar field φ and vector fields $\mathbf{E}$, $\mathbf{B}$ are independent of angles

$$\varphi(\mathbf{r}) = \varphi(r), \quad \mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r, \quad \mathbf{B}(\mathbf{r}) = B_r(r)\,\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{r}|$ and $E_r = E_r(r)$ $B_r = B_r(r)$ are the radial component of $\mathbf{E}$ and $\mathbf{B}$. In this case, we will work with rotationally covariant functions only. This simplification is analogous to the simplification of integration over non-integer dimensional space suggested in [20].

2) *Axially (cylindrical) symmetric case of fractal media*, where the fields $\varphi(r)$ and $\mathbf{E}(r) = E_r(r)\,\mathbf{e}_r$, $\mathbf{B}(r) = B_r(r)\,\mathbf{e}_r$ are also axially symmetric. We assume that z-axis is directed along the axis of symmetry [21].

In [21], the equations of differential operators for non-integer D have been proposed in the following forms, where $m = 1$ and $m = 2$ describe spherically and axially (cylindrical) symmetric cases, respectively.

The divergence in non-integer dimensional space for the vector field $\mathbf{E} = \mathbf{E}(r)$ is

$$\mathrm{Div}^D_r\, \mathbf{E} = \frac{\partial E_r(r)}{\partial r} + \frac{D-m}{r}\, E_r(r). \tag{7}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$\mathrm{Grad}^D_r\, \varphi = \frac{\partial \varphi(r)}{\partial r}\, \mathbf{e}_r. \tag{8}$$

The curl operator for the vector field $\mathbf{E} = \mathbf{E}(r)$ is equal to zero, $\mathrm{Curl}^D_r\, \mathbf{E} = 0$.

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$^S\Delta^D_r \varphi = \mathrm{Div}^D_r\, \mathrm{Grad}^D_r\, \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-m}{r}\,\frac{\partial \varphi}{\partial r}. \tag{9}$$

The vector Laplacian in non-integer dimensional space for the vector field $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ is

$$\begin{aligned} ^V\Delta^D_r \mathbf{E} &= \mathrm{Grad}^D_r\, \mathrm{Div}^D_r\, \mathbf{E} \\ &= \Big(\frac{\partial^2 E_r(r)}{\partial r^2} + \frac{D-m}{r}\,\frac{\partial E_r(r)}{\partial r} - \frac{D-m}{r^2}\, E_r(r)\Big)\,\mathbf{e}_r. \end{aligned} \tag{10}$$

For $D = 3$ equations (7)–(10) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\varphi = \varphi(r)$ and $\mathbf{E}(r) = E_r(r)\,\mathbf{e}_r$.

The suggested operators allow us to reduce D-dimensional vector differentiations (7)–(10) to usual derivatives with respect to $r = |\mathbf{r}|$. As a result, we can reduce partial differential equations for fields in non-integer dimensional space to ordinary differential equations with respect to r. The fractal electrodynamics can be described by operators of integer orders.

We should note that Laplacian, suggested in [18], can be applied only for scalar fields and it cannot be used to describe vector fields $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ and $\mathbf{B} = B_r(r)\,\mathbf{e}_r$ since Stillinger's Laplacian for $D = 3$ is not equal to the usual vector Laplacian for $\mathbb{R}^3$. For the electric and magnetic vector fields $\mathbf{E}$, $\mathbf{B}$ of isotopic fractal case, we should use the vector Laplace operator (10), which is proposed in [21]. Note that the gradient, divergence, curl operator and vector Laplacian are not considered in [18].

In general, the dimension D of the region V_D of fractal media and the dimension d of boundary $S_d = \partial V_D$ of this region are not related by the equation $d = D - 1$, i.e.,

$$\dim(\partial V_D) \neq \dim(V_D) - 1, \tag{11}$$

where $\dim(V_D) = D$. We denote dimension of the boundary $S_d = \partial V_D$ by

$$d = \dim(S_d), \tag{12}$$

and we will use the parameter

$$\alpha_r = D - d, \tag{13}$$

which is a dimension of fractal medium along the radial direction. Using (13), the divergence operator can be written [21] in the form

$$\mathrm{Div}_r^{D,d}\,\mathbf{E} = \pi^{(1-\alpha_r)/2}\,\frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)}\left(\frac{1}{r^{\alpha_r-1}}\frac{\partial E_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\,E_r(r)\right). \tag{14}$$

This is (D,d)-dimensional divergence operator for fractal media with $d \neq D-1$. For $\alpha_r = 1$, i.e. $d = D-1$, equation (14) gives (7).

The gradient for the scalar field $\varphi(\mathbf{r}) = \varphi(r)$ depends on the radial dimension α_r [21] in the form

$$\mathrm{Grad}_r^{D,d}\,\varphi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\,r^{\alpha_r-1}}\,\frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r. \tag{15}$$

Using the operators (15) and (14) for the fields $\varphi = \varphi(r)$ and $\mathbf{E} = E(r)\,\mathbf{e}_r$, we get [21] the scalar and vector Laplace operators for the case $d \neq D-1$ by the equation

$${}^S\Delta_r^{D,d}\varphi = \mathrm{Div}_r^{D,d}\,\mathrm{Grad}_r^{D,d}\,\varphi, \qquad {}^V\Delta_r^{D,d}\mathbf{E} = \mathrm{Grad}_r^{D,d}\,\mathrm{Div}_r^{D,d}\,\mathbf{E}. \tag{16}$$

Then the scalar Laplacian for $d \neq D-1$ for the field $\varphi = \varphi(r)$ is

$${}^S\Delta_r^{D,d}\varphi = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)} \times\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2\varphi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial\varphi}{\partial r}\right), \tag{17}$$

and the vector Laplacian for $d \neq D-1$ for the field $\mathbf{E} = E_r(r)\,\mathbf{e}_r$ is

$${}^V\Delta_r^{D,d}\mathbf{E} = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\left(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2 E_r(r)}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial E_r(r)}{\partial r} - \frac{d\alpha_r}{r^{2\alpha_r}}\,E_r(r)\right)\mathbf{e}_r. \tag{18}$$

Using $\Gamma(1/2) = \sqrt{\pi}$, expressions (17) and (18) with $\alpha_r = 1$, which means $d = D-1$, give (9) and (10).

The vector differential operators (15), (14), (17) and (18), which are suggested in [21], allow us to describe complex fractal media with the boundary dimension of the regions $d \neq D-1$ by the non-integer dimensional space approach.

3. Gauss's law of fractal electrodynamics

Gauss's law may be expressed by the equation

$$\Phi^d_{S_d}(\mathbf{E}) = \frac{1}{\varepsilon_0}\,Q_D(V_D), \tag{19}$$

where ε_0 is the electric constant, $\Phi^d_{S_d}(\mathbf{E})$ is the electric flux through a closed surface S_d with non-integer dimension d enclosing the region V_D with fractal dimension D, such that $\partial V_D = S_d$, and $Q_D(V_D)$ is the total charge enclosed within S_d.

For spherically symmetric case of fractal media, the electric field $\mathbf{E}$ and the charge density $\rho(\mathbf{r})$ are independent of angles

$$\mathbf{E}(\mathbf{r}) = E_r(r)\,\mathbf{e}_r, \quad \rho(\mathbf{r}) = \rho(r),$$

where $E_r = E_r(r)$ is the radial component of $\mathbf{E}$. For this case, the region V_D is a ball with radius R, and the boundary $S_d = \partial V_D$ is a sphere.

The electric flux $\Phi^d_{S_d}(\mathbf{E})$ is defined as a d-dimensional surface integral of the electric field. For spherically symmetric case, the electric flux $\Phi^d_{S_d}(\mathbf{E})$ is defined by the equation

$$\Phi^d_{S_d}(\mathbf{E}) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}\,R^d\;E_r(R). \tag{20}$$

The total charge in region V_D with non-integer dimension $\dim(V_D) = D$ is described by the integral

$$Q_D(V_D) = \int_{V_D} \rho(\mathbf{r})\,d^D\mathbf{r}, \tag{21}$$

where $\mathbf{r}$ is dimensionless vector variable. For a ball region with radius R and the charge density $\rho(\mathbf{r}) = \rho(r)$, the total charge is given by

$$Q_D(V_D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^R dr\,\rho(r)\,r^{D-1}. \tag{22}$$

For the constant charge density $\rho(\mathbf{r}) = \rho_0 = \mathrm{const}$, we have

$$Q_D(V_D) = \rho_0\,V_D = \frac{\pi^{D/2}\rho_0}{\Gamma(D/2+1)}\,R^D. \tag{23}$$

This equation defines the charge of the fractal homogeneous ball with volume V_D. For $D = 3$, equation (23) gives the well-known equation for charge of non-fractal ball $Q_3(V_3) = (4\rho_0\pi/3)R^3$ because $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(x+1) = x\,\Gamma(x)$.

Substitution of (20) and (22) into (19) represents Gauss's law (19) of fractal electrodynamics in the form

$$E_r(R) = \frac{\pi^{(D-d-1)/2}\,\Gamma((d+1)/2)}{\varepsilon_0\,R^d\,\Gamma(D/2)}\int_0^R dr\,\rho(r)\,r^{D-1} \tag{24}$$

for spherically symmetric case. Equation (24) can be represented in the form

$$E_r(R) = \frac{1}{\varepsilon_0\,\varepsilon_{eff}\,R^d}\int_0^R dr\,\rho(r)\,r^{D-1}, \tag{25}$$

where ε_{eff} is effective permittivity of the fractal medium

$$\varepsilon_{eff} = \frac{\Gamma(D/2)}{\pi^{(D-d-1)/2}\,\Gamma((d+1)/2)}. \tag{26}$$

For example, we can consider hollow fractal ball with internal radius R_1 and external radius R_2 with the charge density $\rho(r) =$

$\rho_0 r^\beta$ ($\beta \neq -D$). Then the electric field for the ball point $R_1 < R < R_2$ is

$$E_r(R) = \frac{\pi^{(D-d-1)/2}\,\Gamma((d+1)/2)}{\varepsilon_0\,(D+\beta)\,\Gamma(D/2)}\,\frac{\rho_0(R^{D+\beta} - R_1^{D+\beta})}{R^d}. \quad (27)$$

If $R_1 = 0$, then

$$E_r(R) = \frac{\rho_0\,\pi^{(\alpha_r-1)/2}\,\Gamma((d+1)/2)}{\varepsilon_0\,(D+\beta)\,\Gamma(D/2)}\,R^{\alpha_r+\beta}, \quad (28)$$

where we use the radial dimension $\alpha_r = D - d$. For the constant charge density ($\rho(\mathbf{r}) = \rho_0$), equation (28) gives

$$E_r(R) = \frac{\rho_0\,\pi^{(\alpha_r-1)/2}\,\Gamma((d+1)/2)}{\varepsilon_0\,D\,\Gamma(D/2)}\,R^{\alpha_r}, \quad (29)$$

and the dependence of electric field $E_r(R)$ on the distance R from the center of the ball is defined by the radial dimension only.

4. Ampere's circuital law of fractal electrodynamics

Ampere's circuital law for fractal media states that the circulation $\mathcal{E}^{\gamma}_{L_\gamma}(\mathbf{B})$ of the magnetic field $\mathbf{B}$ around closed γ-dimensional line L_γ is proportional to the total current $I_d(S_d)$, passing through d-dimensional surface S_d (enclosed by L_γ):

$$\mathcal{E}^{\gamma}_{L_\gamma}(\mathbf{B}) = \mu_0\,I_d(S_d), \quad (L_\gamma = \partial S_d). \quad (30)$$

For axially (cylindrical) symmetric case of fractal media, the magnetic field $\mathbf{B}(r) = B_r(r)\,\mathbf{e}_r$ and the total current density $\mathbf{j}(r) = j_r(r)\,\mathbf{e}_r$, are also axially symmetric, where $B_r = B_r(r)$, $j_r = j_r(r)$ are the radial component of $\mathbf{B}$ and $\mathbf{j}$. The region S_d is a circle with radius R, and the boundary $L_\gamma = \partial S_d$ is a circle line.

The circulation $\mathcal{E}^{\gamma}_{L_\gamma}(\mathbf{B})$ of the magnetic field $\mathbf{B}$ is γ-dimensional line integral of the vector field $\mathbf{B}$ around closed γ-dimensional line L_γ. For axially (cylindrical) symmetric case, the circulation $\mathcal{E}^{\gamma}_{L_\gamma}(\mathbf{B})$ of the magnetic field $\mathbf{B}$ is defined by the equation

$$\mathcal{E}^{\gamma}_{L_\gamma}(\mathbf{B}) = \frac{2\pi^{(\gamma+1)/2}}{\Gamma((\gamma+1)/2)}\,R^\gamma\;B_r(R), \quad (31)$$

and the total electric current $I_d(S_d)$ is

$$I_d(S_d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^R dr\,j_r(r)\,r^{d-1}. \quad (32)$$

As a result, Ampere's circuital law (30) of fractal electrodynamics for axially (cylindrical) symmetric case has the form

$$B_r(R) = \mu_0\frac{\pi^{(d-\gamma-1)/2}\,\Gamma((\gamma+1)/2)}{\Gamma(d/2)\,R^\gamma}\int_0^R dr\,j_r(r)\,r^{d-1}, \quad (33)$$

where we can use the second radial dimension $\alpha_2 = d - \gamma$. Equation (33) can be rewritten in the form

$$B_r(R) = \frac{\mu_0\,\mu_{eff}}{R^\gamma}\int_0^R dr\,j_r(r)\,r^{d-1}, \quad (34)$$

where μ_{eff} is the effective permeability

$$\mu_{eff} = \frac{\pi^{(d-\gamma-1)/2}\,\Gamma((\gamma+1)/2)}{\Gamma(d/2)}. \quad (35)$$

For example, we can consider hollow fractal cylindrical current with internal radius R_1 and external radius R_2 with the current density $j_r(r) = j_0\,r^\beta$ ($\beta \neq -d$). Then the magnetic field for the conductor point is

$$B_r(R) = \frac{\mu_0\,j_0\,\pi^{(d-\gamma-1)/2}\,\Gamma((\gamma+1)/2)}{(d+\beta)\,\Gamma(d/2)}\,\frac{R^{d+\beta} - R_1^{d+\beta}}{R^\gamma}. \quad (36)$$

If $R_1 = 0$, then

$$B_r(R) = \frac{\mu_0\,j_0\,\pi^{(\alpha_2-1)/2}\,\Gamma((\gamma+1)/2)}{(d+\beta)\,\Gamma(d/2)}\,R^{\alpha_2+\beta}. \quad (37)$$

For the current density $\beta = d - \gamma$, the magnetic field in the conductor point is constant.

For the constant electric current density ($j(\mathbf{r}) = j_0$), equation (37) gives

$$B_r(R) = \frac{\mu_0\,j_0\,\pi^{(\alpha_2-1)/2}\,\Gamma((\gamma+1)/2)}{d\,\Gamma(d/2)}\,R^{\alpha_2}, \quad (38)$$

and the dependence of magnetic field $B_r(R)$ on distance R from the axis of the cylindrical conductor is defined by the radial dimension $\alpha_2 = d - \gamma$ only.

5. Lorentz invariance and speed of light in fractal electrodynamics

In connection with the use of non-integer dimensions in the fractal electrodynamics, the question about preserving the Lorentz invariance arises for the suggested theory. In this regard, we note that non-integer dimensions are widely used in quantum field theories, including quantum electrodynamics, to remove the ultraviolet divergences. These divergences can be removed by a singular redefinition of the parameters of the theory. This process is called the renormalization that are based on the regularization of the integrals. One of the best renormalization is based on the dimensional regularization [43,44,20]. The main advantage of dimensional regularization is that it preserves not only the Lorentz invariance and the Poincare invariance, but also it preserves the gauge invariance. This is a main motivation to consider the dimensional regularization as a best regularization in quantum field theories. The second advantage of the dimensional regularization is that the integrals do not change the form and the method of calculation has not changed also. As a result, we can state that the non-integer dimensional space approach allows us to preserve the Lorentz invariance, the Poincare invariance and the gauge invariance in the fractal electrodynamics.

There is the second question about speed of light in the fractal electrodynamics. Gauss's law and Ampere's circuital law of fractal electrodynamics allow us to conclude the following: the electric and magnetic fields in non-integer dimensional space can be considered as fields in an effective continuum with the effective permittivity ε_{eff} and the effective permeability μ_{eff}. The analogous conclusion has been suggested in [4]. Therefore, we can assume that the speed of light in fractional electrodynamics can be defined by the equation

$$c_{eff} = \frac{c}{\sqrt{\varepsilon_{eff}\,\mu_{eff}}}, \quad (39)$$

where c is the speed of light in vacuum. Using effective permittivity (26) and effective permeability (35), the effective speed of light (39) is defined by the equation

$$c^1_{eff} = c\sqrt{\frac{\pi^{(D-2d+\gamma)/2}\,\Gamma((d+1)/2)\,\Gamma(d/2)}{\Gamma(D/2)\Gamma((\gamma+1)/2)}}. \quad (40)$$

The refractive index of fractal (or non-integer dimensional space) is

$$n_{eff}^{(1)}(D,d,\gamma)=\sqrt{\frac{\Gamma(D/2)\Gamma((\gamma+1)/2)}{\pi^{(D-2d+\gamma)/2}\,\Gamma((d+1)/2)\,\Gamma(d/2)}}. \quad (41)$$

For non-fractal case, which is defined by integer dimensions $D=3$, $d=2$ and $\gamma=1$, we get $n_{eff}^{(1)}(D,d,\gamma)=1$ and $c_{eff}=c$. It should be noted that fractal case with $d=D-1$ and $\gamma=d-1$ is characterized by $n_{eff}^{(1)}=1$ and $c_{eff}=c$. For $D\in(0;3]$, $\gamma\in(0;1]$ and $d=2$, we have $n_{eff}^{(1)}\geq 1$. For $D=3$, $\gamma=1$ and $d\in[2;3)$, we have $n_{eff}^{(1)}\geq 1$. An anomalous fractal case is $D=3$, $\gamma=1$ and $d\in(0;2)$, when $n_{eff}^{(1)}<1$, that can be considered as an "effective tachyon mode".

Using the wave equations for $\mathbf{E}$, $\mathbf{B}$, and φ, we get the effective speed of light

$$c_{eff}^{(2)}=\frac{c}{n_{eff}^{(2)}(d,\alpha_r)} \quad (42)$$

with the refractive index of fractal (or non-integer dimensional space) in the form

$$n_{eff}^{(2)}(d,\alpha_r)=\sqrt{\frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}}, \quad (43)$$

where $\alpha_r=D-d$. It is easy to see that $n_{eff}^{(2)}(d,\alpha_r)$ does not depend on γ. For $d=2$ and $\alpha_r=1$, we have $n_{eff}^{(2)}(d,\alpha_r)=1$ and $c_{eff}=c$. It is important to note that for all $d\in(0;2]$ and $\alpha_r\in(0;1]$, we have $n_{eff}^{(2)}(d,\alpha_r)\geq 1$ and $c_{eff}\leq c$. As a result, we get that the effective tachyon mode cased by non-integer dimensionality of space does not exist for the propagation of electromagnetic waves in non-integer dimensional spaces.

6. Poisson equation for electric potential in fractal electrodynamics

In fractal electrodynamics, the Poisson equation for electric potential has the form

$${}^{S}\Delta_r^{D,d}\varphi=-\frac{1}{\varepsilon_0}\,\rho, \quad (44)$$

where ${}^{S}\Delta_r^{D,d}$ is the scalar Laplacian. Using (17), the Poisson equation (44) can written as

$$\frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)}\Big(\frac{1}{r^{2\alpha_r-2}}\frac{\partial^2\varphi}{\partial r^2}+\frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\frac{\partial\varphi}{\partial r}\Big) =-\frac{1}{\varepsilon_0}\,\rho. \quad (45)$$

If $\alpha_r=1$ (i.e. $d=D-1$), then we have the equation

$$\frac{\partial^2\varphi(r)}{\partial r^2}+\frac{D-m}{r}\frac{\partial\varphi(r)}{\partial r}=-\frac{1}{\varepsilon_0}\,\rho(r), \quad (46)$$

where $m=1$ and $m=2$ correspond to spherically and axially (cylindrical) symmetric cases, respectively.

Let us consider a uniformly fractal charged infinite circular cylinder of radius R with a constant charge density and non-integer dimension $2<D\leq 3$. Due to the axial symmetry of the charge distribution the potential is also axially symmetric. In the framework of non-integer dimensional space approach, the Poisson equation for scalar potential $\varphi(r)$ created by an infinite circular cylinder has the form

$$\frac{\partial^2\varphi(r)}{\partial r^2}+\frac{D-2}{r}\frac{\partial\varphi(r)}{\partial r}=-\frac{1}{\varepsilon_0}\,\rho(r), \quad (47)$$

where

$$\rho(r)=\begin{cases}\rho_0 & 0<r<R,\\ 0 & r>R.\end{cases} \quad (48)$$

The general solution of equation (47) is

$$\varphi(r)=\begin{cases}C_1+C_2\,r^{3-D}-\dfrac{\rho}{2\,\varepsilon_0\,(D-1)}\,r^2 & 0<r<R,\\ C_3+C_4\,r^{3-D} & r>R,\end{cases} \quad (49)$$

where C_1, C_2, C_3, C_4 are the integration constants, and $2<D\leq 3$. For the case $D=3$, the general solution of equation (47) has the well-known form

$$\varphi(r)=\begin{cases}C_1+C_2\,\ln(r)-\dfrac{\rho}{4\,\varepsilon_0}\,r^2 & 0<r<R,\\ C_3+C_4\,\ln(r) & r>R.\end{cases} \quad (50)$$

The electric field

$$\mathbf{E}(r)=-\,\mathrm{Grad}_r^D\,\varphi(r)=-\frac{\partial\varphi(r)}{\partial r}\,\mathbf{e}_r \quad (51)$$

for potential (49) is

$$\mathbf{E}(r)=\begin{cases}\Big((D-3)\,C_2\,r^{2-D}+\dfrac{\rho}{\varepsilon_0\,(D-1)}\,r\Big)\,\mathbf{e}_r & 0<r<R,\\ (D-3)\,C_4\,r^{2-D}\,\mathbf{e}_r & r>R.\end{cases} \quad (52)$$

The electric field $\mathbf{E}(r)$ must be finite at all points. Therefore we should use $C_2=0$ since $r^{2-D}\to\infty$ for $r\to 0$ for $2<D\leq 3$. The potential can be normalized by the condition $\varphi(0)=0$, then $C_1=0$. Because there are no surface charges, then the electric field (51) at the surface of the cylinder $r=R$ is continuous, i.e. the derivative of the potential should be continuous. The conditions of continuity of the potential and its derivative at $r=R$ give two algebraic equations that allow us to determine the remaining two constants C_3 and C_4 by the equations

$$C_3=-\frac{\rho\,R^2}{2\,\varepsilon_0\,(D-3)},\qquad C_4=\frac{\rho\,R^{D-1}}{\varepsilon_0\,(D-1)\,(D-3)}. \quad (53)$$

As a result, the potential is

$$\varphi(r)=\begin{cases}-\dfrac{\rho}{2\,\varepsilon_0\,(D-1)}\,r^2 & 0<r<R,\\ -\dfrac{\rho\,R^2}{2\,\varepsilon_0\,(D-3)}+\dfrac{\rho\,R^{D-1}}{\varepsilon_0\,(D-1)\,(D-3)}\,r^{3-D} & r>R.\end{cases} \quad (54)$$

Using (51) and (54), the electric field has the form

$$\mathbf{E}(r)=\begin{cases}\dfrac{\rho}{\varepsilon_0\,(D-1)}\,r\,\mathbf{e}_r & 0<r<R,\\ \dfrac{\rho\,R^{D-1}}{\varepsilon_0\,(D-1)}\,r^{2-D}\,\mathbf{e}_r & r>R.\end{cases} \quad (55)$$

For $D=3$, we get the well-known results of non-fractal case.

Equation (55) can be represented in the form

$$\mathbf{E}(r)=\begin{cases}\dfrac{\rho}{2\varepsilon_0\varepsilon_{eff,in}}\,r\,\mathbf{e}_r & 0<r\leq R,\\ \dfrac{1}{2\,\pi\,\varepsilon_0\,\varepsilon_{eff,out}}\,\dfrac{\tau_D}{r^{D-2}}\,\mathbf{e}_r & r>R,\end{cases} \quad (56)$$

where $\varepsilon_{eff,in}$ and $\varepsilon_{eff,out}$ are effective permittivity of fractal medium

$$\varepsilon_{eff,in}=\frac{D-1}{2},\qquad \varepsilon_{eff,out}=\frac{D-1}{2\,\pi^{(3-D)/2}\,\Gamma((D-1)/2)}, \quad (57)$$

and τ_D is the charge per unit length

$$\tau_D=\rho\,V_{D-1}=\rho\,\frac{\pi^{(D-1)/2}R^{D-1}}{\Gamma((D+1)/2)}. \quad (58)$$

For $D=3$, we have $\tau_3=\rho\,\pi\,R^2$ for non-fractal charge cylinder.

The electric field in the fractal homogeneous charged cylinder is analogous to the non-fractal case up to the factor $\varepsilon_{eff,in}$. We have a linear dependence on the distance from the cylinder axis for $0 < r \le R$. Electric field outside the fractal charged cylinder differs from non-fractal case. For $r \ge R$, we have power-law dependence on the distance from the cylinder axis. In addition the electric field outside the cylinder is reduced by the effective permittivity $\varepsilon_{eff,out}$.

7. Fractal stream of charges

Let us find the current density as a function of distance r from the axis of a radially symmetrical parallel stream of charges if the magnetic field $\mathbf{B}(\mathbf{r}) = B_r(r)\mathbf{e}_r$ inside the stream varies as $B_r(r) = b\, r^a$, where a and b are positive constants.

Using the circulation theorem (33) in the form

$$\int_0^R dr\, j_r(r)\, r^{d-1} = \frac{\Gamma(d/2)\, R^{\gamma}}{\mu_0\, \pi^{(d-\gamma-1)/2}\, \Gamma((\gamma+1)/2)}\, B_r(R), \qquad (59)$$

and using $B_r(R) = b\, R^a$, we get the equation

$$\int_0^R dr\, j_r(r)\, r^{d-1} = \frac{b\, \Gamma(d/2)}{\mu_0\, \pi^{(d-\gamma-1)/2}\, \Gamma((\gamma+1)/2)}\, R^{a+\gamma}. \qquad (60)$$

The differentiation of (60) with respect to R gives

$$j_r(r) = \frac{b\, \Gamma(d/2)}{\mu_0\, \pi^{(d-\gamma-1)/2}\, \Gamma((\gamma+1)/2)}\, R^{a+1-\alpha_2}, \qquad (61)$$

where we use the radial dimension $\alpha_2 = d - \gamma$. If $a = \alpha_2 - 1$, then the current density $j_r(r)$ does not depend on distance r from the axis of charge fractal stream.

8. Conclusion

Recently we propose [21,22] the vector calculus for non-integer dimensional space, which includes generalizations of differential operators of first orders (gradient, divergence, curl operators) and the vector Laplace operator. This D-dimensional vector calculus allows us to describe isotropic and anisotropic fractal media in the framework of continuum models with non-integer dimensional spaces. It allows us to extend the scope of possible applications of models with non-integer dimensional spaces. In this paper, we apply vector calculus for non-integer dimensional space suggested in [21] to describe fractal electrodynamics of isotropic fractal distributions of charges, currents, fields and media. Let us note some possible generalization of the approach that is considered in this paper.

We assume that the fractal chains and lattices, which are suggested in [41] (see also [42]), with charged particles in lattice sites can be characterized by fractal distributions of charges and the fractal electrodynamics can be used to describe electromagnetic fields of these chains and lattices.

As we noted in introduction, we should distinguish between two theories: the fractal electrodynamics and the fractional electrodynamics. Fractional theory is based on fractional vector calculus [23]. Note that the fractional-order integrals and derivatives of the Riesz type can be generalized for non-integer dimensional space by continuation in dimension. We assume that it can be used to unite the fractal electrodynamics and the fractional electrodynamics into one general theory. For this aim, we should formulate a fractional-order vector calculus that is based on the Riesz type of derivatives and integrals, and then this calculus should be generalized for non-integer dimensional spaces. This approach will be proposed in the next paper.

The D-dimensional vector calculus suggested in [21] cannot be used to describe anisotropic fractal media and fields. To describe anisotropic fractal media the first order vector differential operators (grad, div, curl) are suggested in [12–17], where these operators are defined as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Generalizations of gradient, divergence, and curl operators to anisotropic fractal case without approximations have been suggested in [22]. Anisotropic fractal media and fields in the framework of continuum models can be described by the D-dimensional vector calculus that is proposed in paper [22]. Anisotropic fractal electrodynamics will be suggested in the next paper.

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