Bosonic string in affine-metric curved space

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The sigma model approach to the closed bosonic string on the affine-metric manifold is considered. The two-loop metric counterterms for the nonlinear two-dimensional sigma-model with affine-metric target manifold are calculated. The correlation of the metric and affine connection is considered as the result of the conformal invariance condition for the nonlinear sigma model. The examples of the nonflat nonriemannian manifolds resulting in the trivial metric beta-function are suggested.

1. Introduction

String theory in a curved space is a consistent quantum theory if the quantum nonlinear two-dimensional sigma model [1,2] is conformally invariant. The conformal invariance requires that the sigma model beta-functions [1,2] be trivial [3]. Since the conformal anomaly of the nonlinear sigma model depends on the geometrical structures (on the background fields) of the curved space (manifold), the beta-function vanishing condition lead to the restrictions on consistent structures (backgrounds fields).

Different geometrical structures can be defined on the manifold [4]. In the bosonic case a metric and a connection structures are used. Riemannian manifolds are considered as a field manifold for usual nonlinear sigma-model [1,2]. The connection strucure of this manifold is uniquely constructed from metric, i.e. the "strong" correlation between the connection and metric structures is postulated. In general case these structures are not correlated [5] and the curved space is nonmetric nonRiemannian manifold. Therefore it was suggested to consider the nonlinear sigma model with nonmetric manifold [7] and to obtain the correlation between the metric and connection structures as the result of the ultraviolet finiteness (or beta-function vanishing) condition for nonlinear sigma model [6].

The sigma model action depends only on the metric structure. Therefore it is surprising that the counterterms of the sigma model with affine-metric manifold differ from counter terms of the sigma model with Riemannian manifold [7]. This difference can not be reduced to the metric redefenition caused by infinitesimal coordinate transformation [2] or to the nonlinear renormalization of the quantum fields [8]. In the paper [7] the counter terms are calculated for conventional sigma model without assuming a metric connection for the geodesic line equation in covariant background field method. In this approach the connection of the sigma model equation of motion is a metric connection and we must use a manifolds with two different connection structures and the metric structure. Therefore it seems more natural for the nonlinear sigma model with nonmetric manifold to consider both the geodesic line equation and the sigma model equation with nonmetric connection (i.e. not derived derived from the metric). It leads to a generalization of the usual sigma model which describes the

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string propagation in affine-metric curved space. String motion on the nonmetric (affine-metric) manifold can be considered as the motion of the string subjected to the dissipative forces. In order to see it we discuss a relationship between the geometrical structures of the manifold and the equation of motion.

The equation of motion for the particle subjected to the force $Q^i(q, u)$ has the form

$$du^i / dt - Q^i(q, u) = 0, \tag{1}$$

where q^i are the coordinates and $u^i = dq^i / dt$ $(i = 1, ..., n)$. We suggest that eq. (1) are invariant under general coordinate transformations and that for simplicity $Q^i(q, u)$ are the gomogeneous functions of second power of u. It is known that the local Lagrange function exists and eq. (1) can be derived from least action principle if and only if the Helmholtz conditions are satisfied. In this case there are matrix multipliers [9,10] such that the eq. (1) become Euler–Lagrange equation. The spesial case $Q^i(q, u) = -[^i{}_{kl}]u^k u^l$, where $[^i{}_{kl}]$ is a Christoffel symbol, the n-dimensional curved space is Riemannian manifold and eq. (1) defines the usual one-dimensional nonlinear sigma model. On the other hand it is known that Lagrange function uniquely defines the metric structure on the $(n + 1)$-dimensional configurational space [11]. That is the equation of motion derived from least action principle is equivalent to the geodesic line equation on metric manifold. The connection structure can be naturally defined on the metric manifold as Christoffel symbols. As the result the motion of the system subjected to potential forces is equivalent to the free motion of the test particle on the metric (Riemann, Finsler or Kawaguchi) manifold, i.e. manifold which connection and metric structures are correlated.

If the Helmholtz conditions are not satisfied, the equation of motion (1) can be represented as the particle motion subjected to dissipative forces Q^i_d on the metric manifold with metric structure defined by the Lagrangian

$$\frac{du^i}{dt} - Q^i_p(q, u) - Q^i_d(q, u) = -(g^{-1})^{ij} D_j L(q, u) - Q^i_d(q, u) = 0, \tag{2}$$

where D_j is the Euler–Lagrange operator, $L(q, u)$ the Lagrange function and $g_{ij}(q, u)$ the matrix multiplier [9]. The dissipative force for the one-dimensional sigma model with affine-metric field manifold is defined by the connection defect $Q^i_d = -D^i{}_{kl}(q)u^k u^l$. If the free motion of the test particle on the manifold are defined by eq. (2) then this manifold is nonmetric. This manifold usually called generalized path space [12] and allow naturally to define connection structure which coefficients are $\Gamma^i{}_{kl}(q, u) = (-1/2)(\partial^2 Q^i/\partial u^k \partial u^l)$. In the generalized path space the connection structure is not correlated with the metric structure of this space. As the result the motion of the systems subjected to dissipative forces on the metric manifold is equivalent to the free motion of the test particle on the nonmetric (generalized path) manifold. Note that the equation of motion and the geodesic line equation in the nonmetrical manifold can be derived from Sedov variational principle [13] which is the generalization of the least action principle.

The affine-metric manifold [5] (path space with metric [18]) is a simple example of the generalized path space with a metric structure. That is the consistent approach to the nonlinear sigma model with affine-metric target manifold lead to a generalization of the usual one-dimensional sigma model which represents a particle subject to dissipative forces. Analogously we have that *the motion of the string in affine-metric curved space is equivalent to the motion of the string subjected to dissipative forces on Riemannian manifold [16].* For this reason the consistent theory of the bosonic string in the curved affine-metric space is a quantum dissipative theory. Note that the dissipative models in fundamental interactions theories are discussed in [23–27,16].

The quantum description of the dissipative systems without well-known ambiguities [20,9,10,24,16], without nonassociative violation of the canonical commutation relations [21] and beyond the sphere of quantum kinetics is suggested in [14–16]. This description uses Sedov variational principle in the phase space to generalize the canonical quantization. The suggested quantization does not violate Heisenberg algebra because it generalizes the canonical quantization by introducing the operator of the nonholonomic quantities in addition to the usual associative operators of the momentum, coordinate and holonomic functions. The generalization of the von Neumann equation was derived from the dissipative Liouville equation [17,15] contrary to usual heuristical and therefore ambiguous generalization [22,23].

In ref. [15] the conformal anomaly of the energy momentum tensor trace for closed bosonic string on the affine-metric manifold is considered and it is proved from the conformal invariance that metric and dilaton beta-functions of the sigma model with affine-metric field manifold must be trivial as usual [3].

In the present paper the two-loop ultraviolet metric counterterms and beta-function for the two-dimensional nonlinear sigma-model with affine-metric field manifold are calculated. The correlation between the connection and the metric structures on the manifold are derived from the beta-function vanishing condition.

2. One-loop and two-loop calculations

Let us consider now the closed bosonic string in curved space–time [19]. The world sheet swept out by the string is described by the map $X(x)$ from two-dimensional parameter space N into n-dimensional space–time manifold M, i.e., $X(x) : N \rightarrow M$. The two-dimensional parameter is $x = (\tau, \sigma)$ and the map $X(x)$ is given by space–time coordinates $X^k(x)$. The classical equation of motion for the closed bosonic string in the n-dimensional affine-metric curved space–time has the form

$$\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu X^i + \Gamma^i{}_{kl}(X) \partial_\mu X^k \sqrt{g} g^{\mu\nu} \partial_\nu X^l = 0, \tag{3}$$

where $g^{\mu\nu}(x)$ is the two-dimensional metric tensor; $\Gamma^i{}_{kl}(X)$ the affine connection, which can be represented in the form $[^i{}_{kl}] + D^i{}_{kl}$; $[^i{}_{kl}]$ is the Christoffel symbol for the metric $G_{ij}(X)$; $D_{ikl}(X)$ is a connection defect tensor which can be written in the form [5]

$$D^i{}_{kl}(X) = (-1/2)G^{ij}(K_{jlk} + K_{jkl} - K_{klj}) + 2Q_{(kl)}{}^i + Q^i{}_{kl}, \tag{4}$$

where $K_{kli} = \nabla_i G_{kl}$ is nonmetricity tensor and $Q^i{}_{kl}$ is torsion tensor. The equation of motion (3) is an equation of the two-dimensional geodesic flow on the affine-metric manifold (the two-dimensional analogue of the geodesic line). It is well known that this equation can not be derived from the least action principle. Note that the Riemannian geodesic flow ($D^i{}_{kl} = 0$) can be derived from this variational principle with the Lagrangian defined by

$$L(X) = (1/2)G_{kl}(X)\partial_\mu X^k \sqrt{g} g^{\mu\nu} \partial_\nu X^l. \tag{5}$$

The affine-metric geodesic flow equation (3) can be derived from the Sedov variational principle [13] if the variation of the nonholonomic functional has the form

$$\delta \widetilde{W} = \int d^2x \, \delta W = -\int d^2x \, D_{ikl}(X) \, \partial_\mu X^k \sqrt{g} g^{\mu\nu} \partial_\nu X^l \, \delta X^i. \tag{6}$$

The holonomic and nonholonomic functionals define a closed bosonic string propagating in the affine-metric curved space–time or in the presence of dissipative and nondissipative background fields.

In loop calculation we use the generating functional for connected Green functions in the phase-space path-integral form for nonhamiltonian (dissipative) systems suggested in [14–16]. This generating functional is written in the form

$$Z(J,g) = -i \ln \int DX DP \exp i \int d^2x [P_k(dX^k/d\tau) - H + W + (i/2)\Omega + K(J)], \tag{7}$$

where $K(J)$ is the source term; Ω is defined in the Appendix (eq. (17)) and $\hbar = 1$. To perform the calculation of the on-shell ultraviolet behavior in one- and two-loop order for sigma model we use the affine-metric covariant background field expansion in normal coordinates [18,7] and new generating functional $Z(X_0, g, J)$. The covariant background field method [2,14] in the phase space is defined by the usual expansion of the coordinates

$X^k(x)$ only. Note that the background field method can be considered as conservative model approximation for the quantum dissipative models. The generating functional $Z(X_0, g, J)$ is defined by

$$\exp i Z(X_0, g, J) = \int D\xi DP \exp i \int d^2x \left(P_k \frac{d}{d\tau} X^k - H + W + (i/2)\Omega + J_k \xi^k \right), \tag{8}$$

where $X = X(X_0, \xi)$; $X_0^i(x)$ is the solution of classical equation of motion; $\xi^k(x)$ the covariant field which is the tangent vector to the affine-metric geodesic line containing X_0^k and X^k.

We produce the the Hamiltonian, nonholonomic functional and omega function in the conformal gauge as a power series in the field $\xi^k(x)$:

$$H = -(1/2)G^{kl}(X)P_k P_l - (1/2)G_{kl}(X)X'^k X'^l, \tag{9}$$

$$W = (1/2)\Delta_1{}^{kl}P_k P_l + (1/2)\Delta^2{}_{kl}X'^k X'^l; \quad \Omega = 2D^k(X)P_k, \tag{10}$$

where $X^i = X^i(X_0, \xi)$; $D^k(X) \equiv D^k{}_{ij}(X)G^{ij}(X)$; $X'^k \equiv (dX^k)/(d\sigma)$; P_k is the canonical momentum. The background field expansions of the Δ-operators are written in the form

$$\Delta_1{}^{kl} = 2D_i{}^{kl}(X_0)\xi^i + O(\xi^2); \quad \Delta^2{}_{kl} = -2D_{ikl}(X_0)\xi^i + O(\xi^2). \tag{11}$$

To obtain all of the one- and two-loop counterterms we need to expand Lagrangian, nonholonomic functional and omega function to fourth order in the quantum fields $\xi^a(x)$. The functional integral of $Z(X_0, g, J)$ over momentum P is the Gaussian integral. It is easy to derive the path integral form for the generating functional:

$$Z(X_0, g, J) = -i \ln \int D\xi \exp i \int d^2x \, A(X(X_0, \xi)). \tag{12}$$

The full expression of $A(X)$ is complicated. Therefore let us consider terms of $A(X)$ which give the nontrivial simple poles two-loop metric divergences only:

$$A(X_0, \xi) = (1/2)\partial_\mu \xi^a \partial_\mu \xi^a + A_{abk}\partial_\mu X_0^k \xi^a \partial_\mu \xi^b + B_{abkl}\xi^a \xi^b \partial_\mu X_0^k \partial_\mu X_0^l + J_{abc}\xi^a \partial_\mu \xi^b \partial_\mu \xi^c + C_{abc}\partial_\mu X_0^k \xi^a \xi^b \partial_\mu \xi^c$$

$$+ L_{abcd}\xi^a \xi^b \partial_\mu \xi^c \partial_\mu \xi^d + E_{abcdp}\partial_\mu X_0^p \xi^a \xi^b \xi^c \partial_\mu \xi^d + F_{abcd}\xi^a \xi^b \partial_\mu \xi^c \kappa^{\mu\nu}\partial_\nu \xi^d,$$

where

$$A_{abk} = [G_{kj;i} + D_{i(jk)} - (1/2)G_{ij;k}]e_a^i e_b^j; \quad J_{abc} = [(1/2)G_{jk;i} + (1/3)D_{i(jk)}]e_a^i e_b^j e_c^k,$$

$$B_{abkl} = [(1/2)\widehat{R}_{kijl} + (1/4)G_{kl;ij} + (1/8)G_{pi;k}G_{pj;l} - (1/2)G_{pj;k}G_{lp;i} + (1/2)D_{i(kl);j}) - (1/2)D_{i(lp)}G_{pj;k}]e_a^i e_b^j,$$

$$C_{abcl} = [(2/3)\widehat{R}_{(k/ij/l)} + (1/2)G_{kl;ij} - (1/2)(G_{pk;i} + (2/3)D_{i(pk)})G_{pj;l} + (2/3)D_{i(kl);j}]e_a^i e_b^j e_c^k,$$

$$E_{ijklp} = [(5/36)G_{n(l/;i}\widehat{R}_{njk/p)} + (1/4)\widehat{R}_{lijp;k} + (1/6)G_{lp;ijk} - (1/4)D_{i(nk);j}G_{nl;p} + (1/6)\widehat{R}_{n(ij)p}D_{k(nl)}$$

$$+ (1/6)\widehat{R}_{n(ij)l}D_{k(np)} + (1/4)D_{(i/lp;/jk)}]e_a^i e_b^j e_c^k e_d^l,$$

$$L_{abcd} = [(1/6)\widehat{R}_{kijl} + (1/4)G_{kl;ij} + (1/4)D_{ikl;j}]e_a^i e_b^j e_c^k e_d^l; \quad F_{ijkl} = 2D_{i(nk)}D_{j(nl)}e_a^i e_b^j e_c^k e_d^l.$$

In the conformal gauge kappa tensor has the form $\kappa^{\mu\nu} = (\kappa^{\tau\tau}, \kappa^{\tau\sigma}, \kappa^{\sigma\sigma}) = (-1, 0, 0)$. We use the following notations:

$$\widehat{R}^i{}_{jkl} = R^i{}_{jkl} + 2\widehat{\nabla}_{[l/}Q^i{}_{j/k]} + 2Q^n{}_{j[k/}Q^i{}_{n/l]}; \quad R^i{}_{jkl} = 2\partial_{[k/}\Gamma^i{}_{j/l]} + 2\Gamma^n{}_{j[l/}\Gamma^i{}_{n/k]},$$
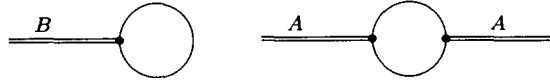
Fig. 1. (a), (b).

$$\widehat{\nabla}_k A_i = \nabla_k A_i + Q^n{}_{ki} A_n = \partial_k A^i - \Gamma^n{}_{(ki)} A_n = A_{i;k}; \quad G_{ij;k} = K_{ijk} + 2Q_{(i/k/j)},$$

$$B_{[n/m} T_{/k]l} = (1/2)(B_{nm} T_{kl} - B_{km} T_{nl}); \quad B_{(j/k/l)} = (1/2)(B_{jkl} + B_{lkj})$$

and $\Gamma^i{}_{(kl)}$ is the symmetry part of the affine connection. The terms of $A(X_0, \xi)$ are usual [1,2] if and only if both the nonmetricity tensor K_{ijl} and he symmetry part of torsion $Q_{(jk)i}$ are equal to zero.

Note that in the expression $A(X_0, \xi)$ we take into account the additional nonmetric terms caused by the following. It is known that propagator of the quantum fields $\xi^k(x)$ is not standard. Therefore we introduce an n-bein $e_k^a(X)$ and define $\xi^a(x) = e_k^a \xi^k(x)$, where $\widehat{\nabla}_k e_l^a = 0$. After this modification the kinetic terms become $\widehat{\nabla}_\mu \xi^a \widehat{\nabla}_\mu \xi^a$, where $\widehat{\nabla}_\mu \xi^a = \partial_\mu \xi^a + \widehat{A}^a{}_{bc} e_k^b \partial_\mu X_0^k \xi^c$. This mixed covariant derivative for the affine-metric manifold M and the Minkowski space N involves the Schouten–Vranceanu connection [29] $\widehat{A}_{abc}$, which is equal to the Ricci rotation coefficient [30] and the object $\omega^a_{kc} \equiv \widehat{A}^a_{bc} e_k^b$ is spin connection [2] on the Riemannian manifold. Note in addition to diagrams of [7] we take into account the diagrams whose external background field lines involve the Schouten–Vranceanu connection. This diagrams must not cancel [15] in contrary to the usual nonlinear sigma-model [2] and give the tensor contribution. It caused by the relation $\widehat{A}_{(a/b/c)} = (-1/2)(K_{ijl} + 2Q_{(i/l/j)}) e_a^i e_c^j e_b^l$.

The irreducible one-loop diagrams (figs. 1a, 1b) produce the following simple poles divergences:

$$(1a) = -(\mu^{2\varepsilon}/4\pi\varepsilon) B_{aakl} \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(1b) = (\mu^{2\varepsilon}/8\pi\varepsilon) A_{[ab]k} A_{[ab]l} \partial_\mu X_0^k \partial_\mu X_0^l,$$

The nontrivial simple poles ultraviolet two-loop divergences are caused by the graphs of figs. 2–6. The two-loop simple poles divergences of these graphs are the following:

$$(2a) = (\mu^{2\varepsilon}/16\pi^2\varepsilon) C_{(ab)ck} C_{a[bc]l} \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(2b) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)(J_{c(ab)} - J_{a(bc)})(C_{(ab)cl;k} + D_{n(ka)} C_{(nb)cl} + D_{n(kb)} C_{(an)cl} + D_{n(kc)} C_{(ab)nl}) \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(2c) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)(J_{a(bc);l} - J_{c(ab);l} + D_{n(la)} J_{n(bc)} + D_{n(lb)} J_{a(nc)} + D_{n(lc)} J_{a(bn)} - D_{n(lc)} J_{n(ba)} - D_{n(lb)} J_{c(na)}$$
$$- D_{n(la)} J_{c(bn)})(J_{a(bc);l} + D_{n(ka)} J_{n(bc)} + D_{n(kb)} J_{a(nc)} + D_{n(kc)} J_{a(bn)}) \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(3a) = -(\mu^{2\varepsilon}/16\pi^2\varepsilon)(L_{cc(ab)} + L_{(ab)cc}) B_{(ab)kl} \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(3b) = (3\mu^{2\varepsilon}/32\pi^2\varepsilon) E_{(cca)bk} A_{[ab]l} \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(3c) = -(\mu^{2\varepsilon}/32\pi^2\varepsilon)(L_{cc(ab)} + L_{(ab)cc})(A_{(ab)k;l} + D_{n(ka)} A_{nbl} + D_{n(kb)} A_{anl}) \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(3d) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)((-1/2) F_{cc(ab)} + (f_1 + (1/2)) F_{(ab)cc}) B_{(ab)kl} \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(3e) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)((f_1 + 1/2) F_{cc(ab)} - (1/2) F_{(ab)cc})(A_{(ab)k;l} + D_{n(ka)} A_{nbl} + D_{n(kb)} A_{anl}) \partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(4a) = -(\mu^{2\varepsilon}/8\pi^2\varepsilon) L_{(ab)(cd)} A_{[ac]k} A^\mu_{[bd]l} \partial_\mu X_0^k \partial_\mu X_0^l,$$
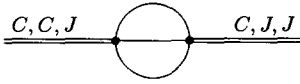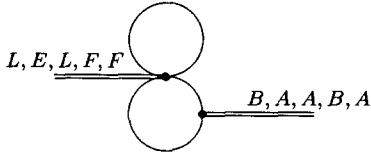
Fig. 2. (a), (b), (c).

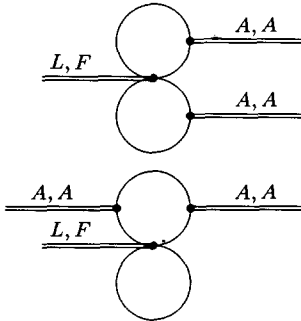
Fig. 3. (a), (b), (c), (d), (e).
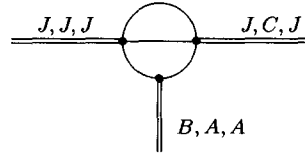

Fig. 4. (a), (b), (c), (d).
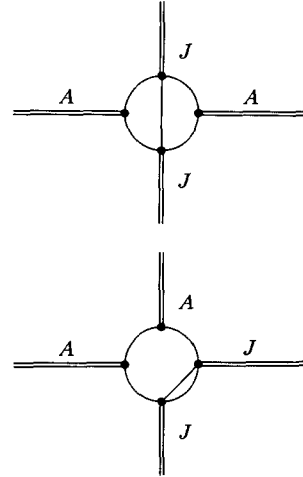

Fig. 5. (a), (b), (c).

Fig. 6. (a), (b).

$$(4b) = (\mu^{2\varepsilon}/48\pi^2\varepsilon)(L_{(ac)dd} - 2L_{dd(ac)})A_{[ab]k}A_{[cb]l}\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(4c,d) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)((1/3)A_{[ab]k}F_{(ac)dd} - (f_1 + (4/3))F_{dd(ac)}A^\mu_{[cd]k})A_{[ab]l}\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(5a) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)((3/2)J_{d(bc)}B_{adkl} + J_{a(bd)}B_{cdkl} - 2J_{b(ad)}B_{cdkl} + +2J_{d(ac)}B_{bdkl})J_{a(bc)}\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(5b) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)(2J_{d(bc)}A_{[ad]k} + (-2)J_{b(cd)}A_{[ad]k} + 2J_{a(bd)}A_{cdk})C_{(ab)cl}\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(5c) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)((3/2)J_{abd}J_{cbd} + J_{bda}J_{bdc} + (-2)J_{bda}J_{dbc})(A_{(ab)k;l} + D_{n(ka)}A_{ncl} + D_{n(kc)}A_{anl})\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(6a) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)((5/3)J_{abd}J_{cbd} - (28/3)J_{bda}J_{cbd} - 4J_{bda}J_{dbc} + 6J_{bda}J_{bdc})A_{[as]k}A_{[cs]l}\partial_\mu X_0^k \partial_\mu X_0^l,$$

$$(6b) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)(J_{bap}J_{dcp} - 2J_{pab}J_{dcp} + (-1/2)J_{pab}J_{pcd} + 2J_{abp}J_{dcp})A_{[ca]k}A_{[bd]l}\partial_\mu X_0^k \partial_\mu X_0^l,$$

where $B_{ac;c}T_a = B_{ac;d}T_b G^{ab}G^{cd}$.

The divergent integrals are calculated using the dimensional regularization (in $n = 2 - 2\varepsilon$ dimensions) with minimal subtraction and general prescription for contraction of the two-dimensional $\kappa^{\mu\nu}$ tensor [15] $\kappa^{\mu\nu}\eta_{\mu\nu} = f(n)$ where $f(n) = 1 + f_1\varepsilon + O(\varepsilon^2)$ and $\eta_{\mu\nu}$ is two-dimensional Minkowski metric. The different prescriptions may correspond to the different renormalization schemes and thus their results should be related through redefenition of the couplings by analogy to the two-dimensional nonlinear sigma-model with Wess–Zumino term [28]. To distinguish between infrared and ultraviolet divergences we introduce an auxilliary mass term [31].

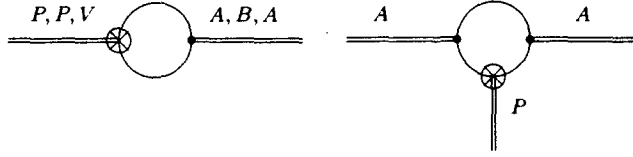The two-loop simple poles divergences caused by one-loop counterterms are derivable from
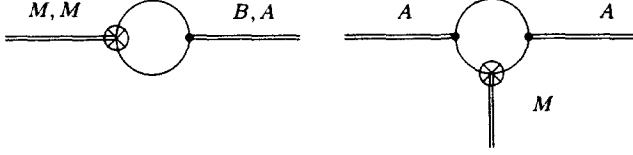
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Fig. 7. (a), (b), (c), (d).

Fig. 8. (a), (b), (c).

$$\Delta L^{(1)} = \frac{\mu^{2\varepsilon}}{4\pi\varepsilon}(P_{ab}\partial_\mu\xi^a\partial_\mu\xi^b + V_{abk}\partial_\mu X_0^k\xi^a\partial_\mu\xi^b + \mu^2 M_{ab}\xi^a\xi^b),$$ (13)

where

$$P_{ab} = [-B_{ccij} + (1/2)A_{[cd]i}A_{[cd]j}]e_a^i e_b^j$$

$$V_{abk} = [2P_{kj;i} - P_{nj}G_{ni;k}]e_a^i e_b^j; \quad M_{ab} = [(-1/6)\widehat{R}_{(i/nn/j)} - G_{in;nj} - G_{ik;l}G_{kl;j}]e_a^i e_b^j.$$

The simple poles divergent part of the graphs (figs. 7, 8) are

$$(7a) = -(\mu^{2\varepsilon}/32\pi^2\varepsilon)P_{(ab)}(A_{(ab)k;l} + D_{n(ka)}A_{ncl} + D_{n(kc)}A_{anl})\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(7b) = -(\mu^{2\varepsilon}/16\pi^2\varepsilon)P_{(ab)}B_{(ab)kl}\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(7c) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)V_{[ab]k}A_{[ab]l}\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(8a) = (\mu^{2\varepsilon}/16\pi^2\varepsilon)M_{(ab)}B_{(ab)kl}\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(8b) = (\mu^{2\varepsilon}/32\pi^2\varepsilon)M_{(ab)}(A_{(ab)k;l} + D_{n(ka)}A_{ncl} + D_{n(kc)}A_{anl})\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(7d) = (2\mu^{2\varepsilon}/48\pi^2\varepsilon)P_{(ab)}A_{[ac]k}A_{[bc]l}\partial_\mu X_0^k\partial_\mu X_0^l,$$

$$(8c) = -(\mu^{2\varepsilon}/48\pi^2\varepsilon)M_{(ab)}A_{[ac]k}A_{[bc]l}\partial_\mu X_0^k\partial_\mu X_0^l.$$

The full expression for the metric beta-function is complicated. Let us consider the special form of the nonmetricity tensor: $K_{ijl} = N_{ijl} = N_{(ijl)}$, where $Q_{(ij)l} = 0$ and $N_{ij(l;k)} = N^n{}_{i(k}N_{l)jn}$. The two-loop metric beta-function [1] for the bosonic nonlinear two-dimensional sigma model with this affine-metric field manifold has the form

$$\beta_{kl}^G = (1/2\pi)[(1/8)N_{nm(k}N_{l)nm} - (1/2)\widehat{R}_{(k/nn/l)}] + (1/4\pi^2)[(1/2)((2/3)\widehat{R}_{(c/(ab)/k)}$$

$$- (1/6)N_{n(c/(a}N_{b)/k)n})((2/3)\widehat{R}_{(c/(ab)/l)} - (2/3)\widehat{R}_{(b/(ac)/k)} + (1/6)N_{n(b/(a}N_{c)/k)n} - (1/6)N_{n(c/(a}N_{b)/l)n})$$

$$+ ((1/2)\widehat{R}_{(k/(ab)/l)} - (1/8)N_{n(a/(k}N_{l)/b)n})((1/6)\widehat{R}_{(a/nn/b)} - (1/6)\widehat{R}_{n/(ab)/n})$$

$$- ((151/72) + (1/2)f_1)N_{nm(a}N_{b)nm})].$$ (14)

This metric beta-function leads to the well-known equation [1,2] on the Riemannian manifold ($K_{ijl} = 0$ and $Q^i{}_{kl} = 0$).

It is easy to see the following ultraviolet finiteness conditions. The one loop and two loop parts of the metric beta-function for the two-dimensional nonlinear sigma-model with affine-metric manifold vanish if the correlation between the affine connection and the metric structures on the manifold M is given by

$$\nabla_l G_{ij} = N_{ijl} = N_{(ijl)}; \quad Q_{(ij)l} = 0; \quad \widehat{\nabla}_{(l}N_{k)ij} = N^p_{i(k}N_{l)jp}; \quad \widehat{R}_{(k/(ij)/l)} = (1/4)\, N^p_{(k/(i}N_{j)/l)p}\,.$$

These conditions have not the f_1 dependence and define nonflat space, the i.e. Riemannian curvature tensor is not equal to zero. Note that the part of the metric beta-function from the sigma model action only is zero in all loops if the affine-metric manifold with the nonmetrisity tensor K_{ijl} and torsion tensor $Q^i{}_{kl}$ is defined by

$$\widehat{R}_{kijl} \equiv R_{kijl} - 2\widehat{\nabla}_{[j/}Q_{ki/l]} - 2Q^n{}_{i[l/}Q_{kn/j]} = 0; \quad \widehat{\nabla}_k G_{ij} = K_{ijk} - 2Q_{(ij)k} = 0\,.$$

It is easy to see that this affine-metric manifold is not flat.

Acknowledgement

Appendix

The equation of motion and the geodesic line equation in a nonmetrical manifold can be derived from the Sedov variational principle [13] which is the generalization of the least action principle:

$$\delta S(q) + \delta\widetilde{W}(q) = 0, \tag{15}$$

where S(q) is the holonomic functional called action and $\widetilde{W}(q)$ is the nonholonomic functional (i.e. $\delta\delta'\widetilde{W} \neq \delta'\delta\widetilde{W}$). For eq. (2) the nonholonomic functional has the form

$$\delta\widetilde{W} = \int \mathrm{d}t\, \delta W = \int \mathrm{d}t\, Q^i_a(q,u)g_{ij}\,\delta q^j\,, \tag{16}$$

i.e. nonholonomic functional is defined by the connection defect. Nonholonomic functional W is characterized by the following properties in the phase space: (1) $[W,p_k] = W^q_k$ and $[W,q^k] = -W^k_p$ i.e. the variation of the functional W is defined by $\delta W = W^q_k\delta q^k + W^k_p\delta p_k$. The brackets are the generalized (variational) Poisson brackets [15,16] which are coincide with usual Poisson brackets for the holonomic functions. 2) $J[Z_k, W, Z_l] = J_{kl} \neq 0$ if $k \neq l$ where $J[A,B,C] = [A[BC]] + [B[CA]] + [C[AB]]$; $k = 1,...,2n$ and $Z_i = q^i$ and $Z_{n+i} = p_i$ if $i = 1,..,n$. The Jacobian J_{kl} characterizes the deviation from the condition of integrability. The object W is the nonholonomic object if one of the J_{kl} is not trivial. Note in addition that the classical phase space equation of motion for dissipative systems has the form $\mathrm{d}Z_k/\mathrm{d}t = [Z_k, H - W]$ and Liouville equation for dissipative systems [17,15] has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(q,p,t) = -\Omega(q,p)\,\rho(q,p,t), \quad \text{where} \quad \Omega(q,p) = \sum_{i=1}^{n} J[q^i, W, p_i]\,. \tag{17}$$

References

[1] D. Friedan, Phys. Rev. Lett. 45 (1980) 1057; Ann. Phys. 163 (1985) 318.

[2] L. Alvarez-Gaume, D.Z. Freedman and S. Mukhi, Ann. Phys. 134 (1981) 85.

[3] G.M. Shore, Nucl. Phys. B 286 (1987) 346.

[4] A.M. Vasiliev, Theory of differentially geometrical structures (Moscow St. Univ. Press, 1987).

[5] J.A. Schouten, Math. Zeitschrift 13 (1922) 56;
J.A. Schouten and D.J. Struik, Einfurhung in die neuren Methoden der Differentialgeometrie I (Noordhoff, Groningen-Batavia, 1935).

[6] V.V. Belokurov and V.E. Tarasov, ICTP preprint IC-90-168.

[7] V.V. Belokurov and V.E. Tarasov, Teoreticheskaya i Matematicheskaya Fizika (USSR) 78 (1989) 471; Theor. Math. Phys. (USA) 78 (1989) 334.

[8] P.S. Howe, G. Papadopoulos and K.S. Stelle, Nucl. Phys. B 296 (1988) 26.

[9] P. Havas, Nuovo Cimento. Ser. 10 Suppl. 5 (1957) 363; Bull. Am. Phys. Soc. 1 No.7 (1956) 337.

[10] I.K. Edvards, Amer. J. Phys. 47 (1979) 153.

[11] C. Lanczos, The variational principles of mechanics, 2nd ed. (Toronto, 1962).

[12] J. Douglas, Ann. Math. 29 (1928) 143;
M.S. Knebelman, Amer. J. Math. 51 (1929) 527;
D.D. Kosambi, C.R. Acad. Sci. Paris. 206 (1938) 1538; Math. Z. 37 (1933) 608.

[13] L.I. Sedov, in: Applied mechanics. Proc. 11th Intern. Congr. Appl. Mech. (Munich, 1964) (Springer-Verlag, 1966) pp. 9–19; in: Irreversible aspects of continuum mechanics and transfer of physical characteristics of moving fluids. Proc. IUTAM Symp. (Vienna, 1966) (Springer-Verlag, 1968) pp. 346–358; Z. Angew. Math. Phys. 20 (1969) 643;
L.I. Sedov and A.G. Chipkin, Macroscopic theory principles of gravity and electromagnetism (Moscow, Nauka, 1989).

[14] V.E. Tarasov, Nucl. Phys. Inst. of Moscow St. Univ. preprint 92-33/282.

[15] V.E. Tarasov, Yadernaj Fiz. (Sov. J. Nucl. Phys.) 56 No .11 (1993) 269.

[16] V.E. Tarasov, Teoreticheskaya i Matematicheskaya Fizika (USSR) 1993/94, to be published.

[17] V.H. Steeb, Physica A 95 (1979) 181.

[18] O. Veblen, Proc. Nat. Acad. Sci. USA. 8 (1922) 192;
O. Veblen and T.Y. Thomas, Trans. Amer. Math. Soc. 25 (1923) 551; 26 (1923) 373;
O. Veblen, Invariants of quadratic differential forms (Cambridge, 1927).

[19] C. Lovelace, Phys. Lett. B 135 (1984) 75;
E.S. Fradkin and A.A. Tseytlin, JETP Lett. 41 (1985) 169; Phys. Lett. B 160 (1985) 69; Nucl. Phys. B 261 (1985) 1;
C.M. Hull and P.K. Townsend, Nucl. Phys. B 274 (1986) 349;
S.P. de Alwis, Phys. Rev. D 34 (1986) 3760;
T. Banks, D. Nemeschansky and A. Sen, Nucl. Phys. B 277 (1986) 67;
C.G. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. B 278 (1986) 78;
J. Maharana and G. Veneziano, Nucl. Phys. B 283 (1987) 126.

[20] N.A. Lemos, Phys. Rev. D 24 (1981) 2338.

[21] R.M. Santilli, Hadronic J. 1 (1978) 223, 574, 1279; 2 (1979) 1460; 3 (1980) 854; 6 (1983) 1873.

[22] G. Lindblad, Comm. Math. Phys. 48 (1976) 119;
J. Messer and B. Baumgartner, Z. Physik. B 32 (1978) 103;
H.J. Korsch and H. Steffen, J. Phys. A 20 (1987) 3787;
M. Hensel and H.J. Korsch, J. Phys. A 25 (1992) 2043.

[23] J. Ellis, J.S. Hagelin, D.V. Nanopoulos and M. Srednicki, Nucl. Phys. B 241 (1984) 381.

[24] I. Prigogine, From being to becoming (W.H. Freeman and Company, San Francisco, 1980).

[25] E.P. Wigner, Amer. J. Phys. 31 (1963) 6.

[26] S. Hawking, Commun. Math. Phys. 87 (1982) 395.

[27] C.G. Callan and L. Thorlacius, Nucl. Phys. B 329 (1990) 117.

[28] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B 293 (1987) 385.

[29] J.A. Schouten, Koninkl. Akad. Wetenschappen te Amsterdam. Proc. Sci. 31. No. 3 (1928) 291;
G. Vranceanu, Rend. Acc. Naz. Linc. Ser. 6 (1926) 4 508; J. de Math. 13 (1934) 113.

[30] S. Chandrasekhar, The mathematical theory of black holes (Oxford, 1983).

[31] A.A. Tseytlin, Int. J. Mod. Phys. A 4 (1989) 1257;
V.V. Belokurov and V.E. Tarasov, Vestnik Moskovskogo Universiteta, Fizika (Moscow Univ. Phys. Bulletin) 32 No. 6 (1991) 14.