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Power-law spatial dispersion from fractional Liouville equation

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A microscopic model in the framework of fractional kinetics to describe spatial dispersion of power-law type is suggested. The Liouville equation with the Caputo fractional derivatives is used to obtain the power-law dependence of the absolute permittivity on the wave vector. The fractional differential equations for electrostatic potential in the media with power-law spatial dispersion are derived. The particular solutions of these equations for the electric potential of point charge in this media are considered. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4825144]

I. INTRODUCTION

In the macroscopic description, the spatial dispersion is represented by non-local connection between the electric displacement field **D** and the electric field **E**. The non-locality is caused by the fact that the field **D** at the point **r** in the medium depends on the values of the electric fields **E** not only in a selected point **r**, but also in its neighborhood points **r**′. Spatial dispersion can be described as a dependence of the absolute permittivity tensor of the medium on the wave vector.[1–3] The electric field in the media with spatial dispersion of the power-law type is described in the recent paper.[4] The spatial dispersion is a characteristic property of the plasma-like media. The term "plasma-like media" was introduced by Silin and Rukhadze in the book.[1] The plasma-like medium is characterized by the presence of free charge carriers, creating as they move in the medium, electric and magnetic fields, which significantly distorts the external field and the effect on the motion of the charges themselves.[1–3] The plasma-like media includes a wide class of object such as ionized gas, metals and semiconductors, molecular crystals, and colloidal electrolytes. The spatial dispersion of the media leads to the set of phenomena, such as the rotation of the plane of polarization, anisotropy of cubic crystals, and other.[5–16]

In the microscopic description of the non-local properties of the media can be considered in the framework of models with long-range interactions of particles.[17–19] Equations of motion for particles with the long-range interactions in the continuous limit can give continuum equations with spatial derivatives of non-integer orders.[20–24] The theory of integration and differentiation of non-integer order[25,26] has a long history,[27,28] and it is concerned with the names of famous mathematicians such as Leibniz, Liouville, Riemann, Abel, Riesz, and Weyl. The fractional derivatives and integrals are powerful tools to describe complex properties of media including long-term memory, non-locality of power-law type, and fractality.[18,19,29–36] Using the fractional calculus, we can consider different generalizations of the Liouville equation[19,37–39] that can be used in the fractional kinetics.[40,41]

We suggest to use the fractional Liouville equations to describe fractional kinetics for plasma-like media with the spatial dispersion of power-law type. Recently, a similar topic including diffusion was considered in Refs. 42 and 43, where using the fractional Fokker-Planck equation was proved that even a small degree of fractionality gives a significant change in the dynamics. The Fokker-Planck equations with fractional coordinate derivatives have been suggested in Ref. 44 (see also Ref. 45) to describe chaotic dynamics. It is known that Fokker-Planck equation for phase-space can be derived from the Liouville equation.[46–48] The Fokker-Planck equation with fractional derivatives is obtained from the fractional Liouville equation in Ref. 38. Instead of the fractional Fokker-Planck equation, we apply the fractional Liouville equations to describe plasma-like media with the power-law spatial dispersion.

In this paper, the Liouville equation with the Caputo fractional derivatives is used to obtain the power-law dependence of the absolute permittivity on the wave vector. This allows us to have a microscopic model for the media with the power-law spatial dispersion, which is described in the recent paper.[4] The appropriate fractional differential equations for electric potential are considered and particular solutions of these equations for the potential in the media with power-law spatial dispersion are suggested. The difference between the point charge potential in the media with this type of spatial dispersion and the Coulomb's and Debye's potentials are described.

In the paper, the degree of spatial non-locality that is described by the value of order α of fractional derivative in the Liouville equation is $0 < \alpha \leq 1$. The reason of using this range of parameter is caused by the fact that the Liouville equation describes the law of conservation of probability, which for the fractional case can be written for these parameter values only. Mathematically, the conservation law for the fractional case is based on fractional generalization of Stokes' and Gauss's theorems,[52] which cannot be formulated for $\alpha > 1$ by the properties of the Caputo fractional derivative and fractional Riemann-Liouville integration.

II. FRACTIONAL LIOUVILLE EQUATION

One of the basic principles of statistical mechanics is the conservation of probability in the phase-space.[50,51]

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The Liouville equation is an expression of the principle in a convenient form for the analysis.

Let us consider dynamics of system in the phase space with dimensionless coordinates $(\mathbf{x},\mathbf{p}) = (x_1,...,x_n,p_1,...,p_n)$. The function $\rho(t,\mathbf{x},\mathbf{p})$ describes probability density to find a system in the phase volume $d^n\mathbf{x}d^n\mathbf{p}$. The evolution of $\rho = \rho(t,\mathbf{x},\mathbf{p})$ is described by the Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{p_i}{m} D^1_{x_i}\rho + F_i D^1_{p_i}\rho = 0, \quad (1)$$

where $F_i = F_i(\mathbf{x},\mathbf{p})$ is the force field. Here, and later we mean the sum on the repeated index i from 1 to n. Equation (1) describes the probability conservation for the volume element of the phase space. If ρ is the one-particle reduced distribution function, then the Liouville equation describes collisionless system. Using the fractional calculus, we can consider different generalizations of the Liouville equation[19,37–39,50] that includes derivatives of non-integer orders.[25]

We can consider a fractional generalization of the Liouville equation in the form

$$\frac{\partial \rho}{\partial t} + \frac{p_i}{m} {}^C_0D^{\alpha_i}_{x_i}\rho + F_i\, {}^C_0D^{\beta_i}_{p_i}\rho = 0, \quad (2)$$

where we use dimensionless variables x_i and p_i, $(i = 1,...,n)$. Here, ${}^C_0D^{\alpha}_x$ and ${}^C_0D^{\beta}_x$ are the Caputo fractional derivatives of order α and β (see Appendix A).

We use Caputo fractional derivatives since a consistent formulation of fractional vector calculus, which contains fractional differential and integral vector operations, can be realized for Caputo differentiation and Riemann-Liouville integration only.[52] It allows to prove the correspondent fractional generalizations of the Green's, Stokes', and Gauss's theorems.[52] The main distinguishing feature of the Caputo fractional derivative is the form of the fractional generalization of the Newton-Leibniz formula (see Lemma 2.22 in Ref. 25) in the usual form

$$F(b) - F(a) = {}_aI^{\alpha}_b\, {}^C_aD^{\alpha}_x F(x), \quad (0 < \alpha < 1). \quad (3)$$

Note that relation (3) is not satisfied for $\alpha > 1$. The other feature of the Caputo fractional derivative is that, like the integer order derivative, the Caputo fractional derivative of a constant is zero.

For simplification, we consider the case $\alpha_i = \alpha$, and $\beta_i = 1$ for all $i = 1,...,n$. The fractional Liouville equation is

$$\frac{\partial \rho}{\partial t} + \frac{p_i}{m} {}^C_0D^{\alpha}_{x_i}\rho + F_i D^1_{p_i}\rho = 0 \quad (0 < \alpha \le 1). \quad (4)$$

The Liouville equation with fractional derivatives with respect to coordinates will be used to describe properties of nonlocal media.

III. PERMITTIVITY OF PLASMA-LIKE NONLOCAL MEDIA

In the absence of the force field $(F_i = 0)$, the Liouville Eq. (4) gives

$$\frac{\partial \rho}{\partial t} + \frac{p_i}{m} {}^C_0D^{\alpha}_{x_i}\rho = 0. \quad (5)$$

The solution of this equation is $\rho_0 = \rho(t,\mathbf{x},\mathbf{p})$, which is the distribution function unperturbed by the fields.

For a weak force field, we use the charge distribution function in the form

$$\rho = \rho_0 + \delta\rho, \quad (6)$$

where ρ_0 is the stationary isotropic homogeneous distribution function unperturbed by the fields, and $\delta\rho$ is the change of ρ_0 by the fields. In the linear approximation with respect to field perturbation, we have

$$\frac{\partial \delta\rho}{\partial t} + \frac{p_i}{m} \left({}^C_0D^{\alpha}_{x_i}\delta\rho\right) + F_i D^1_{p_i}\rho_0 = 0. \quad (7)$$

If we consider plasma-like media, then the force $\mathbf{F} = \mathbf{e}_iF_i$ is the Lorentz force

$$\mathbf{F} = q\mathbf{E}(t,\mathbf{x}) + q[\mathbf{v},\mathbf{B}], \quad (8)$$

where q is charge of particle moves with velocity $\mathbf{v} = \mathbf{p}/m$ in the presence of an electric field $\mathbf{E} = \mathbf{e}_iE_i(t,\mathbf{x})$ and a magnetic field $\mathbf{B}$. Here, and late we use the International System of Units (SI).

In an isotropic media, the distribution function depends only on the magnitude of the momentum, $\rho_0 = \rho_0(|\mathbf{p}|)$. For such a function, the direction of the vector $\mathbf{e}_iD^1_{p_i}\rho_0$ is the same as that of $\mathbf{p} = m\mathbf{v}$, and its scalar product with $[\mathbf{v},\mathbf{B}]$ is equal to zero. Therefore, the magnetic field does not affect the distribution function in the linear approximation. As a result, we have

$$\frac{\partial \delta\rho}{\partial t} + \frac{p_i}{m} \left({}^C_0D^{\alpha}_{x_i}\delta\rho\right) + qE_i D^1_{p_i}\rho_0 = 0. \quad (9)$$

We assume that the perturbation (the function $\delta\rho$ and the field $\mathbf{E}$) is proportional to

$$\delta\rho, \mathbf{E} \sim E_{\alpha}[i(\mathbf{k},\mathbf{x})^{\alpha}] \cdot \exp\{-i\omega t\}, \quad (10)$$

where $E_{\alpha}[z]$ is the Mittag-Leffler function[25]

$$E_{\alpha}[z] := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad (z \in \mathbb{C}, \quad \alpha > 0). \quad (11)$$

For $\alpha = 1$, this function is exponent $E_{\alpha}[z] = \exp\{z\}$, and

$$E_{\alpha}[i(\mathbf{k},\mathbf{x})^{\alpha}]\exp\{-i\omega t\} = \exp\{i(\mathbf{k},\mathbf{x}) - i\omega t\}.$$

We take the x-axis along $\mathbf{k}$. Then, $k_x = |\mathbf{k}|$, $(\mathbf{k},\mathbf{v}) = |\mathbf{k}|v_x$ and Eq. (9) give

$$i\left(|\mathbf{k}|^{\alpha}v_x - \omega\right)\delta\rho + q(E_i D^1_{p_i}\rho_0) = 0, \quad (12)$$

where we use (see Lemma 2.23 in Ref. 25)

$${}^C_0D^{\alpha}_x E_{\alpha}[\lambda x^{\alpha}] = \lambda E_{\alpha}[\lambda x^{\alpha}], \quad (\alpha > 0, \quad \lambda \in \mathbb{C}). \quad (13)$$

As a result, we have

$$\delta\rho = -\frac{q(E_i D^1_{p_i}\rho_0)}{i\,(|\mathbf{k}|^\alpha v_x - \omega)}. \qquad (14)$$

In an unperturbed plasma-like media, the charge density is equal zero, since the media is isotropic. The charge density perturbed by the field is

$$\rho_{charge} = q\int \delta\rho\, d^3\mathbf{p} = iq^2\int \frac{(E_i D^1_{p_i}\rho_0)}{|\mathbf{k}|^\alpha v_x - \omega} d^3\mathbf{p}, \qquad (15)$$

where ρ_{charge} is the bound charge density. The electric polarization vector $\mathbf{P}$ is defined by the relations

$$\mathrm{div}\,\mathbf{P} = -\rho_{charge}. \qquad (16)$$

Then,

$$i(\mathbf{k}, \mathbf{P}) = -\rho_{charge}. \qquad (17)$$

The polarization $\mathbf{P}$ defines the electric displacement field $\mathbf{D}$ as $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, where ε_0 is the electric permittivity. Let the field $\mathbf{E}$ be parallel to $\mathbf{k}$. Then, $\mathbf{P}$ be parallel to $\mathbf{k}$, and

$$\mathbf{P} = (\varepsilon_\parallel(|\mathbf{k}|) - \varepsilon_0)\,\mathbf{E}, \qquad (18)$$

where $\varepsilon_\parallel(|\mathbf{k}|)$ is the longitudinal permittivity.

Substitution of (15) and (18) into (17) gives

$$(\varepsilon_\parallel(|\mathbf{k}|) - \varepsilon_0)\,(|\mathbf{k}|, \mathbf{E}) = -q^2\int \frac{E_i D^1_{p_i}\rho_0}{|\mathbf{k}|^\alpha v_x - \omega - i0} d^3\mathbf{p}. \qquad (19)$$

We add to the frequency ω an infinitesimal positive imaginary part $\delta > 0$, i.e., ω is we replaced by $\omega + i\delta$, where $\delta \to 0+$. In this case, the unlimited increase of the field caused by the factor $\exp(\delta t)$ is unimportant as $t \to \infty$, since the causality principle shows that it cannot affect what is observed at finite times t. As a result, we avoid the poles $\omega = |\mathbf{k}|^\alpha p_x/m$ with analogy of usual case (see Sec. 29 in Ref. 49).

Since we take the x-axis along the vector $\mathbf{k}$, then $\mathbf{E} = (E, 0, 0)$, and $(|\mathbf{k}|, \mathbf{E}) = |\mathbf{k}|E_x$, $E_i D^1_{p_i}\rho_0 = E_x D^1_{p_x}\rho_0$. We introduce the function

$$\rho_0(p_x) = \int \rho_0(|\mathbf{p}|) dp_y dp_z. \qquad (20)$$

As a result, the longitudinal permittivity can be calculated by the equation

$$\varepsilon_\parallel(|\mathbf{k}|) = \varepsilon_0 - \frac{q^2}{|\mathbf{k}|}\int \frac{D^1_{p_x}\rho_0(p_x)}{|\mathbf{k}|^\alpha p_x/m - \omega - i0} dp_x. \qquad (21)$$

For isotropic homogeneous case, we can use an equilibrium distribution $\rho_0(p_x)$.

IV. LONGITUDINAL PERMITTIVITY FOR MAXWELL'S DISTRIBUTION

As mentioned in Sec. II, we use the Liouville equation with fractional derivatives with respect to coordinates (4) and derivatives of integer order with respect to momenta for simplification. This means that we consider the local case with respect to momenta, we assume in our model only spatial non-locality. This assumption leads to derivatives of integer order with respect to momenta, and it allows us to use a Maxwellian background. As a result, we can derive the longitudinal permittivity for a Maxwellian distribution. This assumption is sufficient to explain properties of the media with the power-law spatial dispersion, which are described in Ref. 4.

Let us consider a plasma-like medium with the equilibrium Maxwell's distribution

$$\rho_0(p_x) = \frac{N_q}{\sqrt{2\pi m k_B T}} \exp\left(-\frac{p_x^2}{2mk_BT}\right), \qquad (22)$$

where $k_B = 1.38065 \times 10^{-23}\,\mathrm{m^2kg/(s^2K)}$ is the Boltzmann constant. Then,

$$D^1_{p_x}\rho_0(p_x) = -\frac{2p_xN_q}{\sqrt{\pi}\,(2mk_BT)^{3/2}} \exp\left(-\frac{p_x^2}{2mk_BT}\right). \qquad (23)$$

Here, N_q is the particles number density.

We define the variables

$$z = \frac{p_x}{\sqrt{2mk_BT}}, \quad x = \sqrt{\frac{m}{2k_BT}} \cdot \frac{\omega}{|\mathbf{k}|^\alpha}. \qquad (24)$$

Equation (21) can be rewritten in the form

$$\varepsilon_\parallel(|\mathbf{k}|) = \varepsilon_0 + \frac{q^2N_q}{|\mathbf{k}|^{1+\alpha}} \frac{2m}{\sqrt{\pi}(2mk_BT)^{3/2}} \int_{-\infty}^{+\infty} \frac{p_x}{p_x - m\omega/|\mathbf{k}|^\alpha - i0} \times \exp\left(-\frac{p_x^2}{2mk_BT}\right) dp_x. \qquad (25)$$

Using (24), we have

$$\varepsilon_\parallel(|\mathbf{k}|) = \varepsilon_0 + \frac{q^2}{|\mathbf{k}|^{1+\alpha}} \frac{1}{\sqrt{\pi}k_BT} \int_{-\infty}^{+\infty} \frac{z\,e^{-z^2}}{z - x - i0} dz. \qquad (26)$$

Consider the integral of Eq. (26). Using the formula

$$\int_{-\infty}^{+\infty} \frac{f(z)}{z - i0} dz = P.V.\int_{-\infty}^{+\infty} \frac{f(z)}{z} dz + i\pi f(0),$$

and the relations

$$\frac{z\,e^{-z^2}}{z - x} = e^{-z^2} + \frac{xe^{-z^2}}{z - x}, \quad \int_{-\infty}^{+\infty} e^{-z^2}\, dz = \sqrt{\pi},$$

we obtain

$$\int_{-\infty}^{+\infty} \frac{z\,e^{-z^2}}{z - x - i0} dz = \sqrt{\pi} + P.V.\int_{-\infty}^{+\infty} \frac{xe^{-z^2}}{z - x} dz + i\pi x e^{-x^2}. \qquad (27)$$

We obtain two limiting expressions of (27), and therefore (26), for large and small x.

A. The case of small *x*

For $x \ll 1$, we use the variable $y = z - x$. Then,

$$P.V.\int_{-\infty}^{+\infty}\frac{xe^{-z^2}}{z-x}dz = P.V.\int_{-\infty}^{+\infty}\frac{xe^{-(y+x)^2}}{y}dy. \tag{28}$$

Using

$$e^{-(y+x)^2} = e^{-y^2} - 2ye^{-y^2}x + (2y^2-1)e^{-y^2}x^2 + \frac{1}{6}(12y-8y^3)e^{-y^2}x^3 + \dots,$$

we get

$$P.V.\int_{-\infty}^{+\infty} e^{-y^2}\left(\frac{x}{y} - 2x^2 - \frac{x^3}{y} + 2x^3y + 2x^4 - (4/3)y^2x^4 + \dots\right)dy = -2\sqrt{\pi}x^2 + \sqrt{\pi}x^4 + \dots, \tag{29}$$

where we take into account that the integrals of the odd terms in y are zero.

Substitution of (29) and (27) into (26) gives

$$\varepsilon_{\|}(|\mathbf{k}|) = \varepsilon_0 + \frac{q^2N_q}{|\mathbf{k}|^{1+\alpha}}\frac{1}{k_BT}\left(1 - \frac{m\omega^2}{k_BT\,|\mathbf{k}|^{2\alpha}} + \frac{m^2\omega^4}{4k_B^2T^2\,|\mathbf{k}|^{4\alpha}} + \dots\right) \quad (0 < \alpha \le 1). \tag{30}$$

As a result, we have

$$\varepsilon_{\|}(|\mathbf{k}|) = \varepsilon_0 + \frac{q^2N_q}{k_BT\,|\mathbf{k}|^{1+\alpha}} - \frac{q^2N_qm\omega^2}{k_B^2T^2\,|\mathbf{k}|^{3\alpha+1}} + \frac{q^2N_qm^2\omega^4}{4k_B^3T^3\,|\mathbf{k}|^{5\alpha+1}} + \dots \quad (0 < \alpha \le 1). \tag{31}$$

The imaginary part of the permittivity is relatively small (not exponentially small), in this case because of the smallness of the phase volume in which the condition $|\mathbf{k}|^{\alpha}p_x/m - \omega = 0$ is satisfied. (See comments about the imaginary part in Sec. III.)

The Debye radius of screening is equal to

$$r_D = \sqrt{\frac{\varepsilon_0 k_BT}{N_q q^2}}. \tag{32}$$

The Langmuir frequency for charged particle is

$$\Omega_L = \sqrt{\frac{N_q q^2}{m\varepsilon_0}}. \tag{33}$$

Then, the variable (24) can be represented in the form

$$x = \frac{1}{r_D\,\Omega_L\,\sqrt{2}}\cdot\frac{\omega}{|\mathbf{k}|^{\alpha}}.$$

Note that $\mathbf{k}$, $\mathbf{r}$, and x_i are dimensionless variable.

Using the Debye radius (32) and the Langmuir frequency (33), we rewrite (31) in the form

$$\varepsilon_{\|}(|\mathbf{k}|) \approx \varepsilon_0 + \varepsilon_0\frac{1}{r_D^2\,|\mathbf{k}|^{1+\alpha}} - \varepsilon_0\frac{\omega^2}{r_D^4\,\Omega_L^2\,|\mathbf{k}|^{3\alpha+1}} + \varepsilon_0\frac{\omega^4}{4r_D^6\,\Omega_L^4\,|\mathbf{k}|^{5\alpha+1}} \quad (0 < \alpha \le 1). \tag{34}$$

Using (34), we can derive an equation for the scalar potentials of electric field.

B. The case of large *x*

For $x \gg 1$, we write

$$P.V.\int_{-\infty}^{+\infty}\frac{xe^{-z^2}}{z-x}dz = -\int_{-\infty}^{+\infty}\frac{e^{-z^2}}{1-z/x}dz = -\int_{-\infty}^{+\infty}e^{-z^2}\left(1 + \sum_{s=1}^{\infty}\left(\frac{z}{x}\right)^s\right)dz. \tag{35}$$

Integrals of the odd terms are zero. Then,

$$P.V.\int_{-\infty}^{+\infty}\frac{xe^{-z^2}}{z-x}dz = -\sqrt{\pi} - \frac{\sqrt{\pi}}{2x^2} - \frac{3\sqrt{\pi}}{4x^4} - \dots \quad (x \gg 1). \tag{36}$$

Substituting (36) and (27) into (26), we get

$$\varepsilon_{\|}(|\mathbf{k}|) = 1 - \frac{q^2N_q}{\varepsilon_0|\mathbf{k}|^{1+\alpha}}\frac{1}{k_BT}\left(\frac{k_BT}{m\omega^2}\,|\mathbf{k}|^{2\alpha} + \frac{3k_B^2T^2}{m^2\omega^4}\,|\mathbf{k}|^{4\alpha} + \dots\right) \quad (0 < \alpha \le 1). \tag{37}$$

The imaginary part of $\varepsilon_{\|}(|\mathbf{k}|)$ is exponentially small, since in a Maxwell's distribution only an exponentially small part of the charged particles have the velocity $v_x = \omega/|\mathbf{k}| \gg v_T = \sqrt{k_BT/m}$, where v_T is the average velocity of charged particles.

As a result, we have

$$\varepsilon_{\|}(|\mathbf{k}|) = \varepsilon_0 - \frac{q^2N_q}{m\omega^2}\,|\mathbf{k}|^{\alpha-1} - \frac{3q^2N_qk_BT}{m^2\omega^4}\,|\mathbf{k}|^{3\alpha-1} + \dots \quad (0 < \alpha \le 1). \tag{38}$$

Using the Debye radius (32) and the Langmuir frequency (33), we rewrite (38) in the form

$$\varepsilon_{\|}(|\mathbf{k}|) \approx \varepsilon_0 - \varepsilon_0\frac{\Omega_L^2}{\omega^2}\,|\mathbf{k}|^{\alpha-1} - \varepsilon_0\frac{3r_D^2\Omega_L^4}{\omega^4}\,|\mathbf{k}|^{3\alpha-1} \quad (0 < \alpha \le 1). \tag{39}$$

Using Eqs. (34) and (39), we can obtain the scalar potentials of electric field in power-law nonlocal media, and then describes the difference of these potentials from the well-known Coulomb's and Debye's potentials.

V. SCALAR POTENTIAL OF ELECTRIC FIELD IN NONLOCAL MEDIA

In the case of a static external field sources can create an inhomogeneous electric field $\mathbf{E}(t,\mathbf{r}) = \mathbf{E}(\mathbf{r})$. The electric field in the medium to be a potential

$$\mathbf{E}(\mathbf{r}) = -\text{grad}\,\Phi(\mathbf{r}), \tag{40}$$

where $\Phi(\mathbf{r})$ is a scalar potential of electric field.

Let us consider the 3-dimensional Fourier transform

$$\mathbf{E}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{+i(\mathbf{k},\mathbf{r})}\,\mathbf{E}(\mathbf{k})\,d^3\mathbf{k},$$
$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{+i(\mathbf{k},\mathbf{r})}\,\Phi_{\mathbf{k}}\,d^3\mathbf{k}. \tag{41}$$

The relation (40) gives

$$\mathbf{E}(\mathbf{k}) = -i\mathbf{k}\,\Phi_{\mathbf{k}}. \tag{42}$$

Substituting (42) into the Maxwell equation

$$i(\mathbf{k},\mathbf{E}(\omega,\mathbf{k}))\,\varepsilon(\mathbf{k}) = \rho(\omega,\mathbf{k}), \tag{43}$$

we obtain

$$|\mathbf{k}|^2\,\varepsilon_{\parallel}(|\mathbf{k}|)\,\Phi_{\mathbf{k}} = \rho_{\mathbf{k}}, \tag{44}$$

where $\rho_{\mathbf{k}} = \rho(0,\mathbf{k})$. Note that Eq. (44) does not contain the transverse permittivity $\varepsilon_{\perp}(|\mathbf{k}|)$.

When the field source in the medium is the resting point charge, the charge density is described by delta-distribution

$$\rho(\mathbf{r}) = Q\,\delta^{(3)}(\mathbf{r}), \tag{45}$$

where we have assumed that the charge is at the beginning of the coordinate system. Therefore, the electrostatic potential of the point charge in the isotropic medium according to Eq. (44) has the form

$$\Phi(\mathbf{r}) = \frac{Q}{(2\pi)^3}\int_{\mathbb{R}^3} e^{+i(\mathbf{k},\mathbf{r}-\mathbf{r}_0)}\,\frac{1}{|\mathbf{k}|^2\,\varepsilon_{\parallel}(|\mathbf{k}|)}\,d^3\mathbf{k}. \tag{46}$$

The electric potential (46) created by a point charge Q at a distance $|\mathbf{r}|$ from the charge.

A. The case of the Coulomb potential

If we consider only the first term in Eq. (34), then

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0,$$

where ε_0 is the vacuum permittivity ($\varepsilon_0 \approx 8.854\times 10^{-12} Fm^{-1}$). Substituting $\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0$ into (44), we obtain

$$|\mathbf{k}|^2\,\Phi_{\mathbf{k}} = \frac{1}{\varepsilon_0}\rho_{\mathbf{k}}. \tag{47}$$

The inverse Fourier transform of (47) gives

$$\Delta\Phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r}), \tag{48}$$

where Δ is the 3-dimensional Laplacian for which we have

$$\mathcal{F}[\Delta f(\mathbf{r})](\mathbf{k}) = -|\mathbf{k}|^2\,\mathcal{F}[f(\mathbf{r})](\mathbf{k}). \tag{49}$$

As a result, the electrostatic potential of the point charge (45) is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}|}. \tag{50}$$

This is the Coulomb's form of the potential.

B. The case of the first two terms in Eq. (34) with α=1

If we consider only the first two terms in Eq. (34) with $\alpha = 1$, then

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0\left(1 + \frac{1}{r_D^2|\mathbf{k}|^2}\right). \tag{51}$$

Substituting (51) into (44), we obtain

$$\left(|\mathbf{k}|^2 + \frac{1}{r_D^2}\right)\Phi_{\mathbf{k}} = \frac{1}{\varepsilon_0}\rho_{\mathbf{k}}. \tag{52}$$

The inverse Fourier transform of (47) gives

$$\Delta\Phi(\mathbf{r}) - \frac{1}{r_D^2}\Phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{r}). \tag{53}$$

As a result, we get the screened potential of the point charge (45) in the Debye's form

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}|}\cdot\exp\left(-\frac{|\mathbf{r}|}{r_D}\right), \tag{54}$$

where r_D is the Debye radius of screening. It is easy to see that the Debye's potential differs from the Coulomb's potential by factor $C_D(|\mathbf{r}|) = \exp(-|\mathbf{r}|/r_D)$. Debye's sphere is a region with Debye's radius, in which there is an influence of charges, and outside of which charges are screened.

C. The case of the first two terms in Eq. (34) with $\alpha \neq 1$

If we consider only the first two terms in Eq. (34) with $\alpha \neq 1$, then the longitudinal permittivity $\varepsilon_{\parallel}(|\mathbf{k}|)$ is

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0\left(1 + \frac{1}{r_D^2\,|\mathbf{k}|^{\alpha+1}}\right). \tag{55}$$

Substituting (55) into (44), we obtain

$$\left(|\mathbf{k}|^2 + \frac{1}{r_D^2}\,|\mathbf{k}|^{1-\alpha}\right)\Phi_{\mathbf{k}} = \frac{1}{\varepsilon_0}\rho_{\mathbf{k}}. \tag{56}$$

The inverse Fourier transform of (56) gives

$$-\Delta\Phi(\mathbf{r}) + \frac{1}{r_D^2}\,(-\Delta)^{(1-\alpha)/2}\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}), \tag{57}$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian in the Riesz form (see Appendix B).

Equation (57) is solvable, and its particular solution (see Appendix C) has the form

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} G_{1-\alpha,2}(\mathbf{r}-\mathbf{r}')\, \rho(\mathbf{r}')\, d^3\mathbf{r}', \qquad (58)$$

where

$$G_{1-\alpha,2}(\mathbf{r}) = \frac{|\mathbf{r}|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty (r_D^{-2}\, \lambda^{1-\alpha} + \lambda^2)^{-1} \lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)\, d\lambda. \qquad (59)$$

The electrostatic potential of the point charge (45) is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}|} \cdot C_{1-\alpha,2}(|\mathbf{r}|) \quad (0 < \alpha \le 1), \qquad (60)$$

where the function

$$C_{1-\alpha,2}(|\mathbf{r}|) = \sqrt{\frac{2|\mathbf{r}|}{\pi}} \int_0^\infty \frac{\lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)}{r_D^{-2}\, \lambda^{1-\alpha} + \lambda^2} d\lambda = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{r_D^{-2}\, \lambda^{1-\alpha} + \lambda^2} d\lambda, \quad (0 < \alpha \le 1) \qquad (61)$$

describes the difference between this potential and the Coulomb's potential (50).

Using Sec. 2.3.1 in the book,[57] we obtain the following asymptotic behavior for $C_{1-\alpha,2}(|\mathbf{r}|)$ with $0 < \alpha \le 1$, when $|\mathbf{r}| \to \infty$

$$C_{1-\alpha,2}(|\mathbf{r}|) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{\lambda^2 + r_D^{-2} \lambda^{1-\alpha}} d\lambda \approx A_0(1-\alpha) \frac{1}{|\mathbf{r}|^{1+\alpha}} + \sum_{k=1}^{\infty} A_k(1-\alpha) \frac{1}{|\mathbf{r}|^{(1+\alpha)(k+1)}}, \qquad (62)$$

where the coefficients A_0 and A_k are defined by the relations

$$A_0(1-\alpha) = \frac{2}{\pi r_D^{-2}}\, \Gamma(1+\alpha) \cos\left(\frac{\pi}{2}\alpha\right), \qquad (63)$$

$$A_k(1-\alpha) = -\frac{2}{\pi r_D^{-2(k+1)}} \int_0^\infty z^{(1+\alpha)(k+1)-1} \sin(z)\, dz. \qquad (64)$$

As a result, we have that generalized non-local properties of plasma-like media deform the Debye's screening such that the exponential decay is replaced by the fractional power-law decay

$$C_{1-\alpha,2}(|\mathbf{r}|) \approx \frac{A_0}{|\mathbf{r}|^{1+\alpha}} \quad (0 < \alpha \le 1). \qquad (65)$$

The electrostatic potential of the point charge in the media with this type of spatial dispersion is

$$\Phi(\mathbf{r}) \approx \frac{A_0}{4\pi\varepsilon_0} \cdot \frac{Q}{|\mathbf{r}|^{2+\alpha}} \quad (0 < \alpha \le 1) \qquad (66)$$

on the long distance $|\mathbf{r}| \gg 1$. Equation (66) demonstrates a fractional non-Debye screening of the electric field in the plasma-like media with spatial dispersion of fractional power-law type.

D. The case of the first three terms in Eq. (34) with $\alpha \neq 1$

If we consider the first three terms in Eq. (34) with $\alpha \neq 1$, then

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0 \left(1 + \frac{1}{r_D^2\, |\mathbf{k}|^{1+\alpha}} - \frac{\omega^2}{r_D^4\, \Omega_L^2\, |\mathbf{k}|^{3\alpha+1}}\right). \qquad (67)$$

Substitution of (67) into (44) gives

$$\left(|\mathbf{k}|^2 + \frac{1}{r_D^2} |\mathbf{k}|^{1-\alpha} - \frac{\omega^2}{r_D^4\, \Omega_L^2} |\mathbf{k}|^{1-3\alpha}\right) \Phi_{\mathbf{k}} = \frac{1}{\varepsilon_0} \rho_{\mathbf{k}}. \qquad (68)$$

The inverse Fourier transform of (68) gives

$$-\Delta\Phi(\mathbf{r}) + \frac{1}{r_D^2} (-\Delta)^{(1-\alpha)/2} \Phi(\mathbf{r}) - \frac{\omega^2}{r_D^4\, \Omega_L^2} (-\Delta)^{(1-3\alpha)/2} \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}), \qquad (69)$$

where we use the fractional Laplacian $(-\Delta)^{\alpha/2}$ (see Appendix B). If $1/3 < \alpha \le 1$, then the operator $(-\Delta)^{(1-3\alpha)/2}$ is the Riesz fractional integral (Riesz potential) of the order $(3\alpha-1)/2$.[26]

Equation (69) is solvable, and its particular solution (see Appendix C) has the form

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} G_{1-\alpha,1-3\alpha,2}(\mathbf{r}-\mathbf{r}')\, \rho(\mathbf{r}')\, d^3\mathbf{r}', \qquad (70)$$

where

$$a_1 = \frac{1}{r_D^2}, \quad a_2 = \frac{\omega^2}{r_D^4\, \Omega_L^2}, \qquad (71)$$

and

$$G_{1-\alpha,1-3\alpha,2}(\mathbf{r}) = \frac{|\mathbf{r}|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty \frac{\lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)}{a_1 \lambda^{1-\alpha} - a_2 \lambda^{1-3\alpha} + \lambda^2} d\lambda. \qquad (72)$$

The electrostatic potential of the point charge (45) is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}|} C_{1-\alpha,1-3\alpha,2}(|\mathbf{r}|), \qquad (73)$$

where $0 < \alpha \le 1$, and the function

$$C_{1-\alpha,1-3\alpha,2}(|\mathbf{r}|) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{a_1 \lambda^{1-\alpha} - a_2 \lambda^{1-3\alpha} + \lambda^2} d\lambda, \quad (0 < \alpha \le 1) \qquad (74)$$

describes the difference between this potential and the Coulomb's potential. As a result, characteristic features of the plasma-like with fractional power-law spatial dispersion are non-integer power-law tails analogous to (62), and a fractional power-law decrease of the electric field in such plasma-like media as (66).

E. The case of the first three terms in Eq. (39) with $\alpha \neq 1$

If we consider the first three terms in Eq. (39) with $\alpha \neq 1$, then

$$\varepsilon_{\parallel}(|\mathbf{k}|) = \varepsilon_0 \left(1 - \frac{\Omega_L^2}{\omega^2} |\mathbf{k}|^{\alpha-1} - \frac{3 r_D^2 \Omega_L^4}{\omega^4} |\mathbf{k}|^{3\alpha-1} \right). \qquad (75)$$

Substitution of (75) into (44) gives

$$\left(|\mathbf{k}|^2 - \frac{\Omega_L^2}{\omega^2} |\mathbf{k}|^{\alpha+1} - \frac{3 r_D^2 \Omega_L^4}{\omega^4} |\mathbf{k}|^{3\alpha+1} \right) \Phi_{\mathbf{k}} = \frac{1}{\varepsilon_0} \rho_{\mathbf{k}}. \qquad (76)$$

The inverse Fourier transform of (76) gives

$$-\Delta\Phi(\mathbf{r}) - \frac{\Omega_L^2}{\omega^2} (-\Delta)^{(\alpha+1)/2} \Phi(\mathbf{r}) - \frac{3 r_D^2 \Omega_L^4}{\omega^4} (-\Delta)^{(3\alpha+1)/2} \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}), \qquad (77)$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian (see Appendix B).

If we consider only the first two terms in Eq. (39) with $\alpha \neq 1$, then we have the fractional differential equation

$$-\Delta\Phi(\mathbf{r}) - \frac{\Omega_L^2}{\omega^2} (-\Delta)^{(\alpha+1)/2} \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}). \qquad (78)$$

The electrostatic potential of the point charge (45) has form

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r}|} \cdot C_{\alpha+1,2}(|\mathbf{r}|). \qquad (79)$$

The function

$$C_{\alpha+1,2}(|\mathbf{r}|) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{\lambda^2 - (\Omega_L^2/\omega^2)\, \lambda^{\alpha+1}} d\lambda \quad (0 < \alpha \leq 1) \qquad (80)$$

describes the difference between this potential and the Coulomb's potential.

Using Sec. 2.3.1 in the book,[57] we obtain the asymptotic behavior for $C_{1+\alpha,2}(|\mathbf{r}|)$ with $0 < \alpha \leq 1$, when $|\mathbf{r}| \to \infty$ in the form

$$C_{1+\alpha,2}(|\mathbf{r}|) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{\lambda^2 - (\Omega_L^2/\omega^2)\lambda^{1+\alpha}} d\lambda \approx A_0(1+\alpha) \frac{1}{|\mathbf{r}|^{1-\alpha}} + \sum_{k=1}^{\infty} A_k(1+\alpha) \frac{1}{|\mathbf{r}|^{(1-\alpha)(k+1)}}, \qquad (81)$$

where the coefficients $A_0(1+\alpha)$ and $A_k(1+\alpha)$ are defined by

$$A_0(1+\alpha) = -\frac{2}{\pi(\Omega_L^2/\omega^2)} \Gamma(1-\alpha) \cos\left(\frac{\pi}{2}\alpha\right), \qquad (82)$$

$$A_k(1+\alpha) = -\frac{2}{\pi(-\Omega_L^2/\omega^2)^{(k+1)}} \int_0^\infty z^{(1-\alpha)(k+1)-1} \sin(z)\, dz. \qquad (83)$$

The generalized non-local properties deform the Debye's screening such that the exponential decay is replaced by the fractional power-law decay

$$C_{1+\alpha,2}(|\mathbf{r}|) \approx \frac{A_0}{|\mathbf{r}|^{1-\alpha}} \quad (0 < \alpha \leq 1). \qquad (84)$$

The correspondent electrostatic potential of the point charge in the media with this type of fractional spatial dispersion is given by

$$\Phi(\mathbf{r}) \approx \frac{A_0}{4\pi\varepsilon_0} \cdot \frac{Q}{|\mathbf{r}|^{2-\alpha}} \quad (0 < \alpha \leq 1) \qquad (85)$$

on the long distance $|\mathbf{r}| \gg 1$. As a result, we have a non-Debye screening of the electric field in the plasma-like media with fractional power-law spatial dispersion.

VI. CONCLUSION

The suggested fractional kinetics of plasma-like media gives a microscopic model for the electrodynamics of continuous media with the power-law spatial dispersion of power-law type that is considered in the recent paper.[4] The fractional kinetics is based on a generalization of the Liouville equations that include the Caputo fractional derivatives.[25] Using the fractional Liouville equation, we obtain the power-law dependence of the absolute permittivity on the wave vector. This dependence leads to fractional differential equations for electrostatic potential that includes Riesz fractional derivatives. Particular solutions of these equations, which describe the electric potential of the point charge in the media with power-law spatial dispersion is suggested.

An interesting problem of considerable practical importance is the link between the plasma-like media with fractional power-law spatial dispersion and the identifiable physical and structural features in the media. The following questions relating to this subject can be formulated. What material conditions must be satisfied for the fractional power-law spatial property to be observed? What are the physical interpretations of this type of behavior? We assume that the link between the plasma-like media with fractional power-law spatial dispersion and the physical and structural features in the media (as described in Sec. 5 of Ref. 56) can be realized by the generalized polarizable point dipoles method with fractional screening in molecular dynamics and Monte Carlo simulations. Using these simulations, the presence of identified defects in the structure of plasma-like media can also be taken into account. To realize this suggested approach, additional investigations are required.

The suggested kinetic models allow us to explain by microscopic point, the properties of the media with the fractional power-law spatial dispersion, which are described in Ref. 4. It can help to find new plasma-like media found among objects such as ionized gas, metals and semiconductors, molecular crystals, and colloidal electrolytes. A characteristic feature of such media is non-integer power-law tails, a fractional power-law decrease of the electric field in such plasma-like media. This feature allows us to find the new materials and media by experiments.

Note that the model of fractional dynamics of plasma-like media with power-law spatial dispersion can be considered as new media with a common or universal behavior in space by analogy with the universal behavior of low-loss dielectrics in time.[53–56] This universality and non-Debye screening of the electric field allows us to assume that these media are important for applications.

APPENDIX A: CAPUTO FRACTIONAL DERIVATIVE

The Caputo fractional derivative ${}_a^C D_x^\alpha$ can be defined for functions belonging to the space $AC^n[a,b]$ of absolutely continuous functions.[25] Let $\alpha > 0$ and let n be given by $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $f(x) \in AC^n[a,b]$, then the Caputo fractional derivatives exist almost everywhere on $[a,b]$. If $\alpha \notin \mathbb{N}$, then

$$({}_a^C D_x^\alpha f)(x) = ({}_a I_x^{n-\alpha} D^n f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x dz \frac{D_z^n f(z)}{(x-z)^{\alpha-n+1}},$$

where $n = [\alpha] + 1$. If $\alpha = n \in \mathbb{N}$, then

$$({}_a^C D_x^\alpha f)(x) = D_x^n f(x).$$

It can be directly verified that the Caputo fractional differentiation of the power functions $(x-a)^\beta$ yields power functions

$${}_a^C D_x^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\beta-\alpha},$$

where $\beta > -1$. In particular, then the Caputo fractional derivatives of a constant C are equal to zero

$${}_a^C D_x^\alpha C = 0.$$

For $k = 0, 1, 2, ..., n-1$, we have

$${}_a^C D_x^\alpha (x-a)^k = 0.$$

The Mittag-Leffler function $E_\alpha[\lambda(x-a)^\alpha]$ is invariant[25] with respect to the Caputo derivatives ${}_a^C D_x^\alpha$, i.e.,

$${}_a^C D_x^\alpha E_\alpha[\lambda(x-a)^\alpha] = \lambda E_\alpha[\lambda(x-a)^\alpha].$$

This means that the Mittag-Leffler function is analogous to the exponential for the Caputo fractional derivative.[25]

APPENDIX B: RIESZ FRACTIONAL DERIVATIVE

For $\alpha > 0$ and "sufficiently good" functions $f(x)$, $x \in \mathbb{R}^n$, the Riesz fractional differentiation is defined[25,26] in terms of the Fourier transform $\mathcal{F}$ by

$$(-\Delta)_x^{\alpha/2} f(x) = \mathcal{F}^{-1}(|\mathbf{k}|^\alpha (\mathcal{F}f)(\mathbf{k})). \qquad \text{(B1)}$$

For $\alpha > 0$, the Riesz fractional derivative $(-\Delta)^{\alpha/2}$ can be defined in the form of the hypersingular integral (Sec. 26 in Ref. 26) by

$$(-\Delta)_x^{\alpha/2} f(x) = \frac{1}{d_n(m,\alpha)} \int_{\mathbb{R}^n} \frac{1}{|z|^{\alpha+n}} (\Delta_z^m f)(z)\, dz,$$

where $m > \alpha$, and $(\Delta_z^m f)(z)$ is a finite difference of order m of a function $f(x)$ with a vector step $z \in \mathbb{R}^n$ and centered at the point $x \in \mathbb{R}^n$

$$(\Delta_z^m f)(z) = \sum_{j=0}^m (-1)^j \frac{m!}{j!\,(m-j)!} f(x - jz).$$

The constant $d_n(m,\alpha)$ is defined by

$$d_n(m,\alpha) = \frac{\pi^{1+n/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(n/2+\alpha/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = \sum_{j=0}^m (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha.$$

Note that the hypersingular integral $(-\Delta)_x^{\alpha/2} f(x)$ does not depend on the choice of $m > \alpha$.

If $f(x)$ belongs to the space of "sufficiently good" functions, then the Fourier transform $\mathcal{F}$ of the Riesz fractional derivative is given by

$$(\mathcal{F}(-\Delta)^{\alpha/2} f)(\mathbf{k}) = |\mathbf{k}|^\alpha (\mathcal{F}f)(\mathbf{k}).$$

This equation is valid for the Lizorkin space[26] and the space $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

APPENDIX C: FRACTIONAL DIFFERENTIAL EQUATION

Let us consider the fractional partial differential equation

$$\sum_k^m a_k (-\Delta)^{\alpha_k/2} \Phi(\mathbf{r}) + a_0 \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}), \qquad \text{(C1)}$$

where $\alpha_m > ... > \alpha_1 > 0$, and $a_k \in \mathbb{R}$ are constants. Here, $(-\Delta)^{\alpha_k/2}$ are the fractional Laplacians in the Riesz form.

We apply the Fourier method for solving fractional Eq. (C1). The Fourier transform of the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by

$$\mathcal{F}[(-\Delta)^{\alpha/2} f(\mathbf{r})](\mathbf{k}) = |\mathbf{k}|^\alpha \mathcal{F}[f(\mathbf{r})](\mathbf{k}). \qquad \text{(C2)}$$

Applying the Fourier transform $\mathcal{F}$ to both sides of (C1) and using (C2), we have

$$(\mathcal{F}\Phi)(\mathbf{k}) = \frac{1}{\varepsilon_0} \left(\sum_{k=1}^m a_k |\mathbf{k}|^{\alpha_k} + a_0 \right)^{-1} (\mathcal{F}\rho)(\mathbf{k}). \qquad \text{(C3)}$$

We define the fractional analog of the Green function[25]

$$G_\alpha(\mathbf{r}) = \mathcal{F}^{-1} \left[\left(\sum_{k=1}^m a_k |\mathbf{k}|^{\alpha_k} + a_0 \right)^{-1} \right](\mathbf{r}) = \int_{\mathbb{R}^3} \left(\sum_{k=1}^m a_k |\mathbf{k}|^{\alpha_k} + a_0 \right)^{-1} e^{+i(\mathbf{k},\mathbf{r})}\, d^3\mathbf{k}, \qquad \text{(C4)}$$

where $\alpha = (\alpha_1, ..., \alpha_m)$—is the multi-index.

The following relation

$$\int_{\mathbb{R}^n} e^{i(\mathbf{k},\mathbf{r})} f(|\mathbf{k}|)\, d^n\mathbf{k} = \frac{(2\pi)^{n/2}}{|\mathbf{r}|^{(n-2)/2}} \int_0^\infty f(\lambda)\, \lambda^{n/2} J_{n/2-1}(\lambda|\mathbf{r}|)\, d\lambda \tag{C5}$$

holds (see Lemma 25.1 of Ref. 26) for any function f such that the integral in the right-hand side of (C5) is convergent. Here, J_ν is the Bessel function of the first kind. As a result, the Fourier transform of a radial function is also a radial function.

Using (C5), the Green function (C4) can be represented (see Theorem 5.22 in Ref. 25) in the form of the one-dimensional integral involving the Bessel function of the first kind $J_{1/2}$

$$G_\alpha(\mathbf{r}) = \frac{|\mathbf{r}|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty \left(\sum_{k=1}^m a_k \lambda^{\alpha_k} + a_0 \right)^{-1} \lambda^{3/2} J_{1/2}(\lambda|\mathbf{r}|)\, d\lambda, \tag{C6}$$

where we use $n = 3$, and $\alpha = (\alpha_1, ..., \alpha_m)$—is the multi-index.

If $\alpha_m > 1$ and $A_m \neq 0$, $A_0 \neq 0$, then Eq. (C1) is solvable.[25] The solution of Eq. (C1) can be represented in the form of the convolution of the functions $G(\mathbf{r})$ and $\rho(\mathbf{r})$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} G_\alpha(\mathbf{r} - \mathbf{r}')\, \rho(\mathbf{r}')\, d^3\mathbf{r}', \tag{C7}$$

where the Green function $G_\alpha(z)$ is defined by (C6).

We can consider fractional partial differential Eq. (C1) with $a_0 = 0$ and $a_1 \neq 0$, when $m \in \mathbb{N}$, $m \geq 1$. If $\alpha_1 < 3$, $\alpha_m > 1, m \geq 1, a_1 \neq 0, a_m \neq 0, \alpha_m > ... > \alpha_1 > 0$, then equation

$$\sum_{k=1}^m a_k (-\Delta)^{\alpha_k/2} \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}) \tag{C8}$$

is solvable (Theorem 5.23 in Ref. 25), and its particular solution is given by

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} G_\alpha(\mathbf{r} - \mathbf{r}')\, \rho(\mathbf{r}')\, d^3\mathbf{r}', \tag{C9}$$

where

$$G_\alpha(\mathbf{r}) = \frac{|\mathbf{r}|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty \left(\sum_{k=1}^m a_k \lambda^{\alpha_k} \right)^{-1} \lambda^{3/2} \sqrt{\frac{2}{\pi z}} \sin(z)\, d\lambda. \tag{C10}$$

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