

Electromagnetic field of fractal distribution of charged particles

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Electric and magnetic fields of fractal distribution of charged particles are considered. The fractional integrals are used to describe fractal distribution. The fractional integrals are considered as approximations of integrals on fractals. Using the fractional generalization of integral Maxwell equation, the simple examples of the fields of homogeneous fractal distribution are considered. The electric dipole and quadrupole moments for fractal distribution are derived. © 2005 American Institute of Physics. [DOI: 10.1063/1.1994787]

I. INTRODUCTION

Derivatives and integrals of fractional order¹ have found many applications in recent studies in physics. The interest in fractional analysis has been growing continually in recent years. Fractional analysis has numerous applications: kinetic theories,²⁻⁴ statistical mechanics,⁵⁻⁷ dynamics in complex media,⁸⁻¹² electromagnetic theory,¹³⁻¹⁶ and many others. The new type of problem has increased rapidly in areas in which the fractal features of a process or the medium impose the necessity of using nontraditional tools in “regular” smooth physical equations. In order to use fractional derivatives and fractional integrals for fractal distribution, we must use some continuous medium model.⁹ We propose to describe the fractal distribution by a fractional continuous medium,⁹ where all characteristics and fields are defined everywhere in the volume but they follow some generalized equations which are derived by using fractional integrals. In many problems the real fractal structure of medium can be disregarded and the fractal distribution can be replaced by some fractional continuous mathematical model. Smoothing of microscopic characteristics over the physically infinitesimal volume transforms the initial fractal distribution into fractional continuous model⁹ that uses the fractional integrals. The order of fractional integral is equal to the fractal dimension of distribution. The fractional integrals allow us to take into account the fractality of the distribution. Fractional integrals are considered as approximations of integrals on fractals.^{17,18} In Ref. 17, the authors proved that integrals on net of fractals can be approximated by fractional integrals. In Ref. 5, we proved that fractional integrals can be considered as integrals over the space with fractional dimension up to numerical factor. In order to prove, we use the formulas of dimensional regularizations.¹⁹

We can consider electric and magnetic fields of fractal distribution of charged particles. Fractal distribution can be described by fractional continuous medium model.^{4,9,11,12} In the general case, the fractal distribution of particles cannot be considered as continuous medium. There are points and domains that have no particles. In Ref. 9, we suggest to consider the fractal distributions as special (fractional) continuous media. We use the procedure of replacement of the

distribution with fractal mass dimension by some continuous model that uses fractional integrals. This procedure is a fractional generalization of Christensen approach.²⁰ Suggested procedure leads to the fractional integration and differentiation to describe fractal distribution. In this paper, we consider the electric and magnetic fields of fractal distribution of charged particles. In Sec. II, the densities of electric charge and current for fractal distribution are considered. In Secs. III and IV, we consider the simple examples of the fields of homogeneous fractal distribution. In Sec. V, we consider the fractional generalization of integral Maxwell equation. In Sec. VI, the examples of electric dipole and quadrupole moments for fractal distribution are considered. Finally, a short conclusion is given in Sec. VII.

II. ELECTRIC CHARGE AND CURRENT DENSITIES

A. Electric charge for fractal distribution

Let us consider a fractal distribution of charged particles. For example, we can assume that charged particles with a constant density are distributed over the fractal. In this case, the number of particles N enclosed in a volume with characteristic size R satisfies the scaling law $N(R) \sim R^D$, whereas for a regular n -dimensional Euclidean object we have $N(R) \sim R^n$.

For charged particles with number density $n(\mathbf{r}, t)$, we have that the charge density can be defined by

$$\rho(\mathbf{r}, t) = qn(\mathbf{r}, t),$$

where q is the charge of a particle (for electron, $q = -e$). The total charge of region W is then given by the integral

$$Q(W) = \int_W \rho(\mathbf{r}, t) dV_3,$$

or $Q(W) = qN(W)$, where $N(W)$ is a number of particles in the region W . The fractional generalization of this equation can be written in the following form:

$$Q(W) = \int_W \rho(\mathbf{r}, t) dV_D,$$

where D is a fractal dimension of the distribution, and dV_D is an element of D -dimensional volume such that

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$$dV_D = c_3(D, \mathbf{r}) dV_3. \quad (1)$$

For the Riesz definition of the fractional integral,¹ the function $c_3(D, \mathbf{r})$ is defined by the relation

$$c_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}|^{D-3}. \quad (2)$$

The initial points of the fractional integral are set to zero. The numerical factor in Eq. (2) has this form in order to derive usual integral in the limit $D \rightarrow (3-0)$. Note that the usual numerical factor $\gamma_3^{-1}(D) = \Gamma(1/2)/2^D \pi^{3/2} \Gamma(D/2)$, which is used in Ref. 1 leads to $\gamma_3^{-1}(3-0) = \Gamma(1/2)/2^3 \pi^{3/2} \Gamma(3/2)$ in the limit $D \rightarrow (3-0)$.

For the Riemann-Liouville fractional integral,¹ the function $c_3(D, \mathbf{r})$ is defined by

$$c_3(D, \mathbf{r}) = \frac{|xyz|^{D/3-1}}{\Gamma^3(D/3)}. \quad (3)$$

Here we use Cartesian's coordinates x , y , and z . In order to have the usual dimensions of the physical values, we can use vector \mathbf{r} , and coordinates x, y, z as dimensionless values.

Note that the interpretation of fractional integration is connected with fractional dimension.^{5,6} This interpretation follows from the well-known formulas for dimensional regularizations.¹⁹ The fractional integral can be considered as an integral in the fractional dimension space up to the numerical factor $\Gamma(D/2)/[2\pi^{D/2}\Gamma(D)]$.

If we consider the ball region $W = \{\mathbf{r}: |\mathbf{r}| \leq R\}$ and the spherically symmetric distribution of charged particles $[n(\mathbf{r}, t) = n(r)]$, then we have

$$N(R) = 4\pi \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_0^R n(r) r^{D-1} dr, \quad Q(R) = qN(R).$$

For the homogeneous $[n(r) = n_0]$ fractal distribution, we get

$$N(R) = 4\pi n_0 \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \frac{R^D}{D} \sim R^D.$$

Fractal distribution of particles is called a homogeneous fractal distribution if the power law $N(R) \sim R^D$ does not depend on the translation of the region. The homogeneity property of the distribution can be formulated in the following form: For all regions W and W' such that the volumes are equal, $V(W) = V(W')$, we have that the numbers of particles in these regions are equal too, $N(W) = N(W')$. Note that the wide class of fractal media satisfies the homogeneous property. In Ref. 9, the continuous medium model for the fractal distribution of particles was suggested. Note that the fractality and homogeneity properties can be realized in the following forms:

- (1) Homogeneity: The local number density of homogeneous fractal distribution is a translation invariant value that has the form $n(\mathbf{r}) = n_0 = \text{const}$.
- (2) Fractality: The number of particles in the ball region W obeys a power-law relation $N_D(W) \sim R^D$, where $D < 3$, R is the radius of the ball.

B. Electric current of fractal distribution

For charged particles with number density $n(\mathbf{r}, t)$ flowing with velocity $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$, the resulting density current $\mathbf{J}(\mathbf{r}, t)$ is given by

$$\mathbf{J}(\mathbf{r}, t) = qn(\mathbf{r}, t)\mathbf{u},$$

where q is the charge of a particle (for electron, $q = -e$).

The electric current is defined as the flux of electric charge. Measuring the field $\mathbf{J}(\mathbf{r}, t)$ passing through a surface $S = \partial W$ gives the current (flux of charge)

$$I(S) = \Phi_J(S) = \int_S (\mathbf{J}, d\mathbf{S}_2),$$

where $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ is the current field vector, $d\mathbf{S}_2 = dS_2 \mathbf{n}$ is a differential unit of area pointing perpendicular to the surface S , and the vector $\mathbf{n} = n_k \mathbf{e}_k$ is a vector of normal. The fractional generalization of this equation for the fractal distribution can be written in the following form:

$$I(S) = \int_S (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_d),$$

where we use

$$dS_d = c_2(d, \mathbf{r}) dS_2, \quad c_2(d, \mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}|^{d-2}. \quad (4)$$

Note that $c_2(2, \mathbf{r}) = 1$ for $d = 2$. In general, the boundary ∂W has the dimension d . In the general case, the dimension d is not equal to 2 and is not equal to $(D-1)$.

C. Charge conservation for fractal distribution

The electric charge has a fundamental property established by numerous experiments: the change of the quantity of charge inside a region W bounded by the surface $S = \partial W$ is always equal to the flux of charge through this surface. This is known as the law of charge conservation. If we denote by $\mathbf{J}(\mathbf{r}, t)$ the electric current density, then charge conservation is written

$$\frac{dQ(W)}{dt} = -I(S),$$

or, in the form

$$\frac{d}{dt} \int_W \rho(\mathbf{r}, t) dV_D = - \oint_{\partial W} (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_d). \quad (5)$$

In particular, when the surface $S = \partial W$ is fixed, we can write

$$\frac{d}{dt} \int_W \rho(\mathbf{r}, t) dV_D = \int_W \frac{\partial \rho(\mathbf{r}, t)}{\partial t} dV_D. \quad (6)$$

Using the fractional generalization of the mathematical Gauss's theorem (see Appendix), we have

$$\oint_{\partial W} (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_d) = \int_W c_3^{-1}(D, \mathbf{r}) \frac{\partial}{\partial x_k} [c_2(d, \mathbf{r}) J_k(\mathbf{r}, t)] dV_D. \quad (7)$$

Substituting the right-hand sides of Eqs. (6) and (7) in Eq. (5), we find the law of charge conservation in differential form

$$c_3(D, \mathbf{r}) \frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \frac{\partial}{\partial x_k} [c_2(d, \mathbf{r}) J_k(\mathbf{r}, t)] = 0.$$

This equation can be considered as a continuity equation for fractal distribution of particles.¹¹

III. ELECTRIC FIELD OF FRACTAL DISTRIBUTION

A. Electric field and Coulomb's law

For a point charge Q at position \mathbf{r}' (i.e., an electric monopole), the electric field at a point \mathbf{r} is defined in MKS by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3},$$

where ϵ_0 is a fundamental constant called the permittivity of free space.

For a continuous stationary distribution $\rho(\mathbf{r}')$ of charge, the electric field \mathbf{E} at a point \mathbf{r} is given in MKS by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_W \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') dV'_3, \quad (8)$$

where ϵ_0 is the permittivity of free space. For Cartesian's coordinates $dV'_3 = dx' dy' dz'$.

The fractional generalization of Eq. (8) for a fractal distribution of charge is given by the equation

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_W \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') dV'_D, \quad (9)$$

where $dV'_D = c_3(D, \mathbf{r}') dV'_3$. Equation (9) can be considered as Coulomb's law written for a fractal stationary distribution of electric charges.

Measuring the electric field passing through a surface $S = \partial W$ gives the electric flux

$$\Phi_E(S) = \int_S (\mathbf{E}, d\mathbf{S}_2),$$

where \mathbf{E} is the electric-field vector, and $d\mathbf{S}_2$ is a differential unit of area pointing perpendicular to the surface S .

B. Gauss's law for fractal distribution

Gauss's law tells us that the total flux $\Phi_E(S)$ of the electric field \mathbf{E} through a closed surface $S = \partial W$ is proportional to the total electric charge $Q(W)$ inside the surface,

$$\Phi_E(\partial W) = \frac{1}{\epsilon_0} Q(W). \quad (10)$$

The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.

For the fractal distribution, Gauss's law states

$$\int_S (\mathbf{E}, d\mathbf{S}_2) = \frac{1}{\epsilon_0} \int_W \rho(\mathbf{r}, t) dV_D \quad (11)$$

in MKS, where $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ is the electric field, and $\rho(\mathbf{r}, t)$ is the charge density, $dV_D = c_3(D, \mathbf{r}) dV_3$, and ϵ_0 is the permittivity of free space.

Gauss's law by itself can be used to find the electric field of a point charge at rest, and the principle of superposition can then be used to find the electric field of an arbitrary fractal charge distribution.

If we consider the stationary spherically symmetric fractal distribution $\rho(\mathbf{r}, t) = \rho(r)$ and the ball region $W = \{\mathbf{r}: |\mathbf{r}| \leq R\}$, then we have

$$Q(W) = 4\pi \int_0^R \rho(r) c_3(D, \mathbf{r}) r^2 dr,$$

where $c_3(D, \mathbf{r})$ is defined in Eq. (2), i.e.,

$$Q(W) = 4\pi \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)} \int_0^R \rho(r) r^{D-1} dr. \quad (12)$$

Using the sphere $S = \{\mathbf{r}: |\mathbf{r}| = R\}$ as a surface $S = \partial W$, we get

$$\Phi_E(\partial W) = 4\pi R^2 E(R). \quad (13)$$

Substituting Eqs. (12) and (13) in Gauss's law (10), we get the equation for electric field. As a result, Gauss's law for fractal distribution with spherical symmetry leads us to the equation for electric field

$$E(R) = \frac{2^{3-D} \Gamma(3/2)}{\epsilon_0 R^2 \Gamma(D/2)} \int_0^R \rho(r) r^{D-1} dr.$$

For example, the electric field of homogeneous [$\rho(\mathbf{r}) = \rho$] spherically symmetric fractal distribution is defined by

$$E(R) = \rho \frac{2^{3-D} \Gamma(3/2)}{\epsilon_0 D \Gamma(D/2)} R^{D-2} \sim R^{D-2}.$$

IV. MAGNETIC FIELD OF FRACTAL DISTRIBUTION

A. Magnetic field and Biot-Savart law

The Biot-Savart law relates magnetic fields to the currents which are their sources. In a similar manner, Coulomb's law relates electric fields to the point charges which are their sources. Finding the magnetic field resulting from a fractal current distribution involves the vector product and is inherently a fractional calculus problem when the distance from the current to the field point is continuously changing.

For a continuous distribution the Biot-Savart law in MKS has the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_W \frac{[\mathbf{J}(\mathbf{r}'), \mathbf{r} - \mathbf{r}']}{|\mathbf{r} - \mathbf{r}'|^3} dV'_3, \quad (14)$$

where $[\]$ is a vector product, \mathbf{J} is the current density, μ_0 is the permeability of free space.

The fractional generalization of Eq. (14) for a fractal distribution in MKS has the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_W \frac{[\mathbf{J}(\mathbf{r}'), \mathbf{r} - \mathbf{r}']}{|\mathbf{r} - \mathbf{r}'|^3} dV'_D. \quad (15)$$

This equation can be considered as Biot-Savart law written for a steady current with fractal distribution of electric charges. The Biot-Savart law (15) can be used to find the magnetic field produced by any fractal distribution of steady currents.

B. Ampere's law for fractal distribution

The magnetic field in space around an electric current is proportional to the electric current which serves as its source, just as the electric field in space is proportional to the charge which serves as its source. In the case of static electric field, the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. The Ampere's law is equivalent to the steady state of the integral Maxwell equation in free space, and relates the spatially varying magnetic field $\mathbf{B}(\mathbf{r})$ to the current density $\mathbf{J}(\mathbf{r})$.

Note that, as mentioned by Lutzen in his article,²¹ Liouville, who was one of pioneers in development of fractional calculus, was inspired by the problem of fundamental force law in Ampere's electrodynamics and used fractional differential equation in that problem.

Let be a closed path around a current. Ampere's law states that the line integral of the magnetic field \mathbf{B} along the closed path L is given in MKS by

$$\oint_L (\mathbf{B}, d\mathbf{l}) = \mu_0 I(S),$$

where $d\mathbf{l}$ is the differential length element, and μ_0 is the permeability of free space. For the fractal distribution of charged particles, we use

$$I(S) = \int_S (\mathbf{J}, d\mathbf{S}_d),$$

where $d\mathbf{S}_d = c_2(d, \mathbf{r}) dS_2$.

If we consider the cylindrically symmetric fractal distribution, we have

$$I(S) = 2\pi \int_0^R J(r) c_2(d, \mathbf{r}) r dr,$$

where $c_2(d, \mathbf{r})$ is defined in Eq. (4), i.e.,

$$I(S) = 4\pi \frac{2^{2-d}}{\Gamma(d/2)} \int_0^R J(r) r^{d-1} dr.$$

Using the circle $L = \partial W = \{\mathbf{r} : |\mathbf{r}| = R\}$, we get

$$\oint_L (\mathbf{B}, d\mathbf{l}) = 2\pi R B(R).$$

As a result, Ampere's law for fractal distribution with cylindrical symmetry leads us to the equation for magnetic field

$$B(R) = \frac{\mu_0 2^{2-d}}{R\Gamma(d/2)} \int_0^R J(r) r^{d-1} dr.$$

For example, the magnetic field $B(r)$ of homogeneous [$J(r) = J_0$] fractal distribution is defined by

$$B(R) = j_0 \frac{\mu_0 2^{2-d}}{d\Gamma(d/2)} R^{d-1} \sim R^{d-1}.$$

V. FRACTIONAL INTEGRAL MAXWELL EQUATIONS

The Maxwell equations are the set of fundamental equations governing electromagnetism (i.e., the behavior of electric and magnetic fields). The equations that can be expressed in integral form are known as Gauss's law, Faraday's law, the absence of magnetic monopoles, and Ampere's law with displacement current. In MKS, these become

$$\oint_S (\mathbf{E}, d\mathbf{S}_2) = \frac{1}{\varepsilon_0} \int_W \rho dV_D,$$

$$\oint_L (\mathbf{E}, d\mathbf{l}_1) = - \frac{\partial}{\partial t} \int_S (\mathbf{B}, d\mathbf{S}_2),$$

$$\oint_S (\mathbf{B}, d\mathbf{S}_2) = 0,$$

$$\oint_L (\mathbf{B}, d\mathbf{l}_1) = \mu_0 \int_S (\mathbf{J}, d\mathbf{S}_d) + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S (\mathbf{E}, d\mathbf{S}_2).$$

Let us consider the special case such that the fields are defined on fractal²² only. The hydrodynamic and thermodynamics fields can be defined in the fractal media.^{4,11} Suppose that the electromagnetic field can be defined on fractal as an approximation of some real case with fractal medium. If the electric field $\mathbf{E}(\mathbf{r})$ and magnetic fields $\mathbf{B}(\mathbf{r})$ can be defined on fractal and does not exist outside of fractal in Euclidian space E^3 , then we must use the fractional generalization of the integral Maxwell equations in the form

$$\oint_S (\mathbf{E}, d\mathbf{S}_d) = \frac{1}{\varepsilon_0} \int_W \rho dV_D,$$

$$\oint_L (\mathbf{E}, d\mathbf{l}_\gamma) = - \frac{\partial}{\partial t} \int_S (\mathbf{B}, d\mathbf{S}_d),$$

$$\oint_S (\mathbf{B}, d\mathbf{S}_d) = 0,$$

$$\oint_L (\mathbf{B}, d\mathbf{l}_\gamma) = \mu_0 \int_S (\mathbf{J}, d\mathbf{S}_d) + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S (\mathbf{E}, d\mathbf{S}_d).$$

These fractional integral equations have unusual properties. Note that fractional integrals are considered as an approximation of integrals on fractals.^{17,18}

Using the fractional generalization of Stokes's and Gauss's theorems (see Appendix), we can rewrite the fractional integral Maxwell equations in the form

$$\int_W c_3^{-1}(D, \mathbf{r}) \operatorname{div}[c_2(d, \mathbf{r}) \mathbf{E}] dV_D = \frac{1}{\epsilon_0} \int_W \rho dV_D,$$

$$\int_S c_2^{-1}(d, \mathbf{r}) (\operatorname{curl}[c_1(\gamma, \mathbf{r}) \mathbf{E}], d\mathbf{S}_d) = - \frac{\partial}{\partial t} \int_S (\mathbf{B}, d\mathbf{S}_d),$$

$$\int_W c_3^{-1}(D, \mathbf{r}) \operatorname{div}[c_2(d, \mathbf{r}) \mathbf{B}] dV_d = 0,$$

$$\int_S c_2^{-1}(d, \mathbf{r}) (\operatorname{curl}[c_1(\gamma, \mathbf{r}) \mathbf{B}], d\mathbf{S}_d)$$

$$= \mu_0 \int_S (\mathbf{J}, d\mathbf{S}_d) + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S (\mathbf{E}, d\mathbf{S}_d).$$

As a result, we have the following differential Maxwell equations:

$$\operatorname{div}[c_2(d, \mathbf{r}) \mathbf{E}] = \frac{1}{\epsilon_0} c_3(D, \mathbf{r}) \rho,$$

$$\operatorname{curl}[c_1(\gamma, \mathbf{r}) \mathbf{E}] = - c_2(d, \mathbf{r}) \frac{\partial}{\partial t} \mathbf{B},$$

$$\operatorname{div}[c_2(d, \mathbf{r}) \mathbf{B}] = 0,$$

$$\operatorname{curl}[c_1(\gamma, \mathbf{r}) \mathbf{B}] = \mu_0 c_2(d, \mathbf{r}) \mathbf{J} + \epsilon_0 \mu_0 c_2(d, \mathbf{r}) \frac{\partial \mathbf{E}}{\partial t}.$$

Note that the law of absence of magnetic monopoles for the fractal leads us to the equation $\operatorname{div}[c_2(d, \mathbf{r}) \mathbf{B}] = 0$. This equation can be rewritten in the form

$$\operatorname{div} \mathbf{B} = - (\mathbf{B}, \operatorname{grad} c_2(d, \mathbf{r})).$$

In the general case ($d \neq 2$), the vector $\operatorname{grad}[c_2(d, \mathbf{r})]$ is not equal to zero and the magnetic field satisfies $\operatorname{div} \mathbf{B} \neq 0$. If $d = 2$, we have $\operatorname{div}(\mathbf{F}) \neq 0$ only for nonsolenoidal field \mathbf{F} . Therefore the magnetic field on the fractal is similar to the nonsolenoidal field. As a result, the magnetic field on fractal can be considered as a field with some "fractional magnetic monopole" $q_m \sim (\mathbf{B}, \nabla c_2)$.

VI. MULTIPOLE MOMENTS FOR FRACTAL DISTRIBUTION

A. Electric multipole expansion

A multipole expansion is a series expansion of the effect produced by a given system in terms of an expansion parameter which becomes small as the distance away from the system increases. Therefore, the leading one of the terms in a multipole expansion are generally the strongest. The first-order behavior of the system at large distances can therefore be obtained from the first terms of this series, which is generally much easier to compute than the general solution. Multipole expansions are most commonly used in problems

involving the gravitational field of mass aggregations, the electric and magnetic fields of charge and current distributions, and the propagation of electromagnetic waves.

To compute one particular case of a multipole expansion, let $\mathbf{R} = X_k \mathbf{e}_k$ be the vector from a fixed reference point to the observation point, $\mathbf{r} = x_k \mathbf{e}_k$ be the vector from the reference point to a point in the body, and $\mathbf{d} = \mathbf{R} - \mathbf{r}$ be the vector from a point in the body to the observation point. The law of cosines then yields

$$d^2 = R^2 + r^2 - 2rR \cos \theta = R^2 \left(1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos \theta \right),$$

where $d = |\mathbf{d}|$, and $\cos \theta = (\mathbf{r}, \mathbf{R}) / (rR)$, so

$$d = R \sqrt{1 + \frac{r^2}{R^2} - 2 \frac{r}{R} \cos \theta}.$$

Now define $\epsilon = r/R$, $x = \cos \theta$, then

$$\frac{1}{d} = \frac{1}{R} (1 - 2\epsilon x + \epsilon^2)^{-1/2}.$$

But $(1 - 2\epsilon x + \epsilon^2)^{-1/2}$ is the generating function for Legendre polynomials $P_n(x)$ as follows:

$$(1 - 2\epsilon x + \epsilon^2)^{-1/2} = \sum_{n=0}^{\infty} \epsilon^n P_n(x),$$

so, we have the equation

$$\frac{1}{d} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^n P_n(\cos \theta).$$

Any physical potential that obeys a $(1/d)$ law can therefore be expressed as a multipole expansion

$$U = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos \theta) \rho(\mathbf{r}) dV_D. \quad (16)$$

The $n=0$ term of this expansion, called the monopole term, can be pulled out by noting that $P_0(x) = 1$, so

$$U = \frac{1}{4\pi\epsilon_0} \frac{1}{R} \int_W \rho(\mathbf{r}) dV_D$$

$$+ \frac{1}{4\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos \theta) \rho(\mathbf{r}) dV_D. \quad (17)$$

The n th term

$$U_n = \frac{1}{4\pi\epsilon_0} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos \theta) \rho(\mathbf{r}) dV_D \quad (18)$$

is commonly named according to the following: n -multipole, 0-monopole, 1-dipole, 2-quadrupole.

B. Electric dipole moment of fractal distribution

An electric multipole expansion is a determination of the voltage U due to a collection of charges obtained by performing a multipole expansion. This corresponds to a series expansion of the charge density $\rho(\mathbf{r})$ in terms of its moments,

normalized by the distance to a point \mathbf{R} far from the charge distribution. In MKS, the electric multipole expansion is given by Eq. (16),

$$U = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int_W r^n P_n(\cos \theta) \rho(\mathbf{r}) dV_D, \quad (19)$$

where $P_n(\cos \theta)$ is a Legendre polynomial and θ is the polar angle, defined such that

$$\cos \theta = (\mathbf{r}, \mathbf{R}) / (|\mathbf{r}| |\mathbf{R}|).$$

The first term arises from $P_0(x)=1$, while all further terms vanish as a result of $P_n(x)$ being a polynomial in x for $n \geq 1$, giving $P_n(0)=0$ for all $n \geq 1$.

If we have

$$Q(W) = \int_W \rho(\mathbf{r}) dV_D = 0,$$

then the $n=0$ term vanishes. Set up the coordinate system so that θ measures the angle from the charge-charge line with the midpoint of this line being the origin. Then the $n=1$ term is given by

$$\begin{aligned} U_1 &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^2} \int_W r P_1(\cos \theta) \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^2} \int_W r \cos \theta \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^2} \int_W \frac{(\mathbf{r}, \mathbf{R})}{R} \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^3} \int_W (\mathbf{r}, \mathbf{R}) \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^3} \left(\mathbf{R}, \int_W \mathbf{r} \rho(\mathbf{r}) dV_D \right). \end{aligned}$$

For a continuous charge distribution, the electric dipole moment is given by

$$\mathbf{p} = \int_W \mathbf{r} \rho(\mathbf{r}) dV_3, \quad (20)$$

where \mathbf{r} points from positive to negative. Defining the dipole moment for the fractal distribution by the equation

$$\mathbf{p}^{(D)} = \int_W \mathbf{r} \rho(\mathbf{r}) dV_D, \quad (21)$$

then gives

$$U_1 = \frac{1}{4\pi\epsilon_0} \frac{(\mathbf{R}, \mathbf{p}^{(D)})}{R^3} = \frac{1}{4\pi\epsilon_0} \frac{p^{(D)} \cos \alpha}{R^2}$$

where $\cos \alpha = (\mathbf{R}, \mathbf{p}^{(D)}) / [p^{(D)} R]$, and $p^{(D)} = \sqrt{[p_x^{(D)}]^2 + [p_y^{(D)}]^2 + [p_z^{(D)}]^2}$.

Let us consider the dipole moment for the fractal distribution by Eq. (21), where we use the Riemann-Liouville fractional integral, and the function $c_3(D, \mathbf{r})$ in the form

$$c_3(D, \mathbf{r}) = \frac{|xyz|^{a-1}}{\Gamma^3(a)}, \quad a = D/3. \quad (22)$$

Let us consider the example of electric dipole moment for the homogeneous $[\rho(\mathbf{r})=\rho]$ fractal distribution of electric charges in the parallelepiped region

$$W = \{(x; y; z): 0 \leq x \leq A, 0 \leq y \leq B, 0 \leq z \leq C\}. \quad (23)$$

In this case, we have Eq. (21) in the form

$$\begin{aligned} p_x^{(D)} &= \frac{\rho}{\Gamma^3(a)} \int_0^A dx \int_0^B dy \int_0^C dz x^a y^{a-1} z^{a-1} \\ &= \frac{\rho(ABC)^a}{\Gamma^3(a)} \frac{A}{a^2(a+1)}. \end{aligned}$$

The electric charge of parallelepiped region (23) is defined by

$$Q(W) = \rho \int_W dV_D = \frac{\rho(ABC)^a}{a^3 \Gamma^3(a)}.$$

Therefore, we have the dipole moments for fractal distribution in parallelepiped in the form

$$\begin{aligned} p_x^{(D)} &= \frac{a}{a+3} Q(W) A, \quad p_y^{(D)} = \frac{a}{a+3} Q(W) B, \\ p_z^{(D)} &= \frac{a}{a+3} Q(W) C, \end{aligned}$$

where we can use $a/(a+1) = D/(D+3)$. As a result, we get

$$p_k^{(D)} = \frac{2D}{D+3} p_k^{(3)}, \quad (24)$$

where $p_k^{(3)}$ are the dipole moments for three-dimensional homogeneous distribution. If we use the following limits $2 < D \leq 3$, then we have

$$0.8 < \frac{2D}{D+3} \leq 1.$$

C. Electric quadrupole moment of fractal distribution

While this is the dominant term for a dipole, there are also higher-order terms in the multipole expansion that become smaller as R becomes large. The electric quadrupole term in MKS is given by

$$\begin{aligned} U_2 &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^3} \int_W r^2 P_2(\cos \theta) \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{R^3} \int_W r^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{2R^3} \int_W r^2 (3 \cos^2 \theta - 1) \rho(\mathbf{r}) dV_D \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{2R^3} \int_W \left[\frac{3}{R^2} (\mathbf{R}, \mathbf{r})^2 - r^2 \right] \rho(\mathbf{r}) dV_D. \end{aligned}$$

The electric quadrupole is the third term in an electric multipole expansion and can be defined in MKS by

$$U_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{2R^3} \sum_{k,l=1}^3 \frac{X_k X_l}{R^2} Q_{kl},$$

where ϵ_0 is the permittivity of free space, R is the distance from the fractal distribution of charges, and Q_{kl} is the electric quadrupole moment, which is a tensor.

The electric quadrupole moment is defined by the equation

$$Q_{kl} = \int_W (3x_k x_l - r^2 \delta_{kl}) \rho(\mathbf{r}) dV_D,$$

where $x_k = x, y, \text{ or } z$. From this definition, it follows that

$$Q_{kl} = Q_{lk} \quad \text{and} \quad \sum_{k=1}^3 Q_{kk} = 0.$$

Therefore, we have $Q_{zz} = -Q_{xx} - Q_{yy}$. In order to compute the values

$$\begin{aligned} Q_{xx}^{(D)} &= \int_W [3x^2 - (x^2 + y^2 + z^2)] \rho(\mathbf{r}) dV_D \\ &= \int_W [2x^2 - y^2 - z^2] \rho(\mathbf{r}) dV_D, \end{aligned}$$

$$\begin{aligned} Q_{yy}^{(D)} &= \int_W [3y^2 - (x^2 + y^2 + z^2)] \rho(\mathbf{r}) dV_D \\ &= \int_W [-x^2 + 2y^2 - z^2] \rho(\mathbf{r}) dV_D, \end{aligned}$$

$$\begin{aligned} Q_{zz}^{(D)} &= \int_W [3z^2 - (x^2 + y^2 + z^2)] \rho(\mathbf{r}) dV_D \\ &= \int_W [-x^2 - y^2 + 2z^2] \rho(\mathbf{r}) dV_D, \end{aligned}$$

we consider the following expression:

$$Q(\alpha, \beta, \gamma) = \int_W [\alpha x^2 + \beta y^2 + \gamma z^2] \rho(\mathbf{r}) dV_D, \quad (25)$$

where we use the Riemann-Liouville fractional integral,¹ and the function $c_3(D, \mathbf{r})$ in the form

$$c_3(D, \mathbf{r}) = \frac{|xyz|^{a-1}}{\Gamma^3(a)}, \quad a = D/3. \quad (26)$$

Using Eq. (25), we have

$$\begin{aligned} Q_{xx}^{(D)} &= Q(2, -1, -1), \quad Q_{yy}^{(D)} = Q(-1, 2, -1), \\ Q_{zz}^{(D)} &= Q(-1, -1, 2). \end{aligned} \quad (27)$$

D. Quadrupole moment of fractal parallelepiped

Let us consider the example of electric quadrupole moment for the homogeneous $[\rho(\mathbf{r}) = \rho]$ fractal distribution of electric charges in the parallelepiped region

$$W = \{(x; y; z): 0 \leq x \leq A, 0 \leq y \leq B, 0 \leq z \leq C\}. \quad (28)$$

If we consider the region W in form (28), then we get

$$Q(\alpha, \beta, \gamma) = \frac{\rho(ABC)^a}{(a+2)a^2\Gamma^3(a)} (\alpha A^2 + \beta B^2 + \gamma C^2).$$

The electric charge of this region W is

$$Q(W) = \rho \int_W dV_D = \frac{\rho(ABC)^a}{a^3\Gamma^3(a)}.$$

Therefore, we have the following equation:

$$Q(\alpha, \beta, \gamma) = \frac{a}{a+2} Q(W) (\alpha A^2 + \beta B^2 + \gamma C^2),$$

where $a = D/3$. If $D = 3$, then we have

$$Q(\alpha, \beta, \gamma) = \frac{1}{3} Q(W) (\alpha A^2 + \beta B^2 + \gamma C^2).$$

As a result, we get electric quadrupole moments $Q_{kk}^{(D)}$ of fractal distribution in the region W ,

$$Q_{kk}^{(D)} = \frac{3D}{D+6} Q_{kk}^{(3)},$$

where $Q_{kk}^{(3)}$ are moments for the usual homogeneous distribution ($D = 3$). By analogy with these equations, we can derive $Q_{kl}^{(D)}$ for the case $k \neq l$. These electric quadrupole moments are

$$Q_{kl}^{(D)} = \frac{4D^2}{(D+3)^2} Q_{kl}^{(3)}, \quad (k \neq l).$$

If we use the following limits $2 < D \leq 3$, then we get the relations

$$0.75 < \frac{3D}{D+6} \leq 1, \quad 0.64 < \frac{4D^2}{(D+3)^2} \leq 1.$$

E. Quadrupole moment of fractal ellipsoid

Let us consider the example of electric quadrupole moment for the homogeneous $[\rho(\mathbf{r}) = \rho]$ fractal distribution in the ellipsoid region W ,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \leq 1. \quad (29)$$

If we consider the region W in the form (29), then we get (25) in the form

$$\begin{aligned} Q(\alpha, \beta, \gamma) &= \frac{\rho(ABC)^a}{(3a+2)\Gamma^3(a)} [\alpha A^2 K_1(a) + \beta B^2 K_2(a) \\ &\quad + \gamma C^2 K_3(a)], \end{aligned}$$

where $a = D/3$, and $K_i(a)$ ($i = 1, 2, 3$) are defined by

$$K_1(a) = L(a + 1, a - 1, 2\pi)L(a - 1, 2a + 1, \pi),$$

$$K_2(a) = L(a - 1, a + 1, 2\pi)L(a - 1, 2a + 1, \pi),$$

$$K_3(a) = L(a - 1, a - 1, 2\pi)L(a + 1, 2a - 1, \pi).$$

Here we use the following function:

$$L(n, m, l) = \frac{2l}{\pi} \int_0^{\pi/2} dx |\cos(x)|^n |\sin(x)|^m$$

$$= \frac{l}{\pi} \frac{\Gamma(n/2 + 1/2)\Gamma(m/2 + 1/2)}{\Gamma(n/2 + m/2 + 1)}.$$

If $D=3$, we obtain

$$Q(\alpha, \beta, \gamma) = \frac{4\pi \rho ABC}{3} \frac{\rho ABC}{5} (\alpha A^2 + \beta B^2 + \gamma C^2), \tag{30}$$

where we use $K_1=K_2=K_3=4\pi/3$. The total charge of this region W is

$$Q(W) = \rho \int_W dV_D = \frac{\rho(ABC)^a 2\Gamma^3(a/2)}{3a\Gamma^3(a)} \frac{1}{\Gamma(3a/2)}. \tag{31}$$

If $D=3$, we have the total charge

$$Q(W) = \rho \int_W dV_3 = \frac{4\pi}{3} \rho ABC. \tag{32}$$

Using Eq. (31), we get the electric quadrupole moments (27) for fractal ellipsoid

$$Q(\alpha, \beta, \gamma) = \frac{a}{3a+2} Q(W) [\alpha A^2 + \beta B^2 + \gamma C^2], \tag{33}$$

where $a=D/3$. If $D=3$, then we have the well-known relation

$$Q(\alpha, \beta, \gamma) = \frac{Q(W)}{5} (\alpha A^2 + \beta B^2 + \gamma C^2).$$

VII. CONCLUSION

In this paper, we have introduced and described the fractional continuous model for the fractal distribution of charged particles. Using the fractional calculus and the fractional continuous model, we have shown that the fractional integrals can be used for calculation of multipole moments of the fractal distribution. The order of fractional integral is equal to the fractal dimension of the distribution.

The fractional continuous models for fractal distribution of particles may have applications in plasma physics. This is due in part to the relatively small numbers of parameters that define a fractal distribution of great complexity and rich structure. The fractional generalization of integral Maxwell equations may have applications in the analysis of electrodynamic problems involving the fractal structures. Therefore, it is interesting to numerically solve the fractional equations for charged fractals. The fractional continuous model can be used to describe dynamics and kinetics of the fractal distribution in the plasma physics. Extension of this model to describe the dynamical properties of fractal distribution by

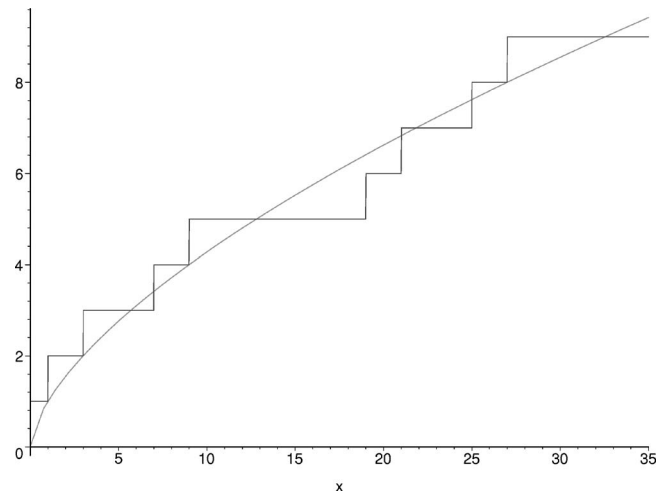


FIG. 1. Charge of fractal distribution in the interval [0; 30].

fractional generalization of magnetohydrodynamics and Vlasov equations is currently under study by the author.

APPENDIX A: FRACTIONAL GAUSS'S THEOREM

In order to realize the representation, we derive the fractional generalization of Gauss's theorem

$$\int_{\partial W} (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_2) = \int_W \text{div}[\mathbf{J}(\mathbf{r}, t)] dV_3,$$

where the vector $\mathbf{J}(\mathbf{r}, t) = J_k \mathbf{e}_k$ is a field, and $\text{div}(\mathbf{J}) = \partial J_k / \partial x_k$. Here and later we mean the sum on the repeated index k from 1 to 3. Using the relation

$$d\mathbf{S}_d = c_2(d, \mathbf{r}) d\mathbf{S}_2, \quad c_2(d, \mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}|^{d-2},$$

we get

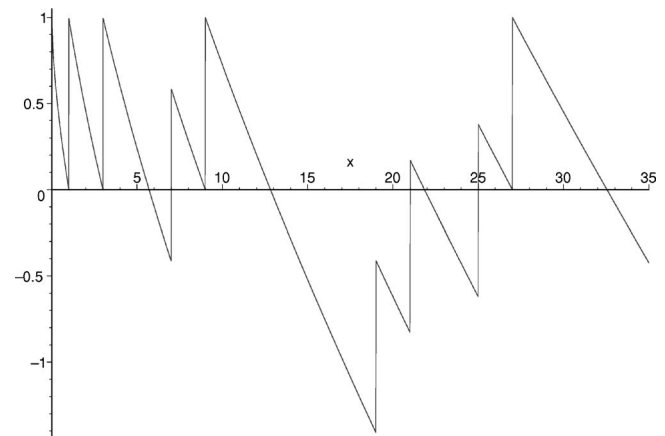


FIG. 2. Difference between the charge of fractal distribution and fractional continuous model.

$$\int_{\partial W} (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_d) = \int_{\partial W} c_2(d, \mathbf{r})(\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_2).$$

Note that we have $c_2(2, \mathbf{r})=1$ for the $d=2$. Using the usual Gauss's theorem, we get

$$\int_{\partial W} c_2(d, \mathbf{r})(\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_2) = \int_W \text{div}[c_2(d, \mathbf{r})\mathbf{J}(\mathbf{r}, t)]dV_3.$$

The relation

$$dV_D = c_3(D, \mathbf{r})dV_3, \quad c_3(D, \mathbf{r}) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}|\mathbf{r}|^{D-3}$$

in the form $dV_3 = c_3^{-1}(D, \mathbf{r})dV_D$ allows us to derive the fractional generalization of Gauss's theorem,

$$\int_{\partial W} (\mathbf{J}(\mathbf{r}, t), d\mathbf{S}_d) = \int_W c_3^{-1}(D, \mathbf{r})\text{div}[c_2(d, \mathbf{r})\mathbf{J}(\mathbf{r}, t)]dV_D.$$

Analogously, we can get the fractional generalization of Stokes's theorem in the form

$$\oint_L (\mathbf{E}, d\mathbf{l}_\gamma) = \int_S c_2^{-1}(d, \mathbf{r})(\text{curl}[c_1(\gamma, \mathbf{r})\mathbf{E}], d\mathbf{S}_d),$$

where

$$c_1(\gamma, \mathbf{r}) = \frac{2^{1-\gamma}\Gamma(1/2)}{\Gamma(\gamma/2)}|\mathbf{r}|^{\gamma-1}.$$

APPENDIX B: CANTOR SET AND FRACTIONAL CONTINUOUS MODEL

In the paper, we mention the difference between the real fractal medium structures and replacing it by a fractional continuous mathematical model. Some quantitative measure of the difference would be helpful. Note that the difference between the real fractal media and fractional continuous medium model has an analog of the difference between the real atomic structure of the media and the usual continuous medium models of these media. In order to have some quantitative measure of the applicability of fractional continuous medium model, we can consider the power law for fractal media. The fractal distribution of charged particles is characterized by the law $Q([0, R]) \sim R^D$.²²

The Cantor set is given by taking the interval $[0; x]$, removing the open middle third, removing the middle third of each of the two remaining pieces, and continuing this procedure *ad infinitum*. The Cantor set is sometimes also called no middle third set.

In Fig. 1, we consider the charge $y=Q([0, x])$ of fractal distribution. The fractal distribution is described by the Cantor set with fractal dimension $D=\ln(2)/\ln(3)$.²² The continuous model for the charge distribution is described by the continuous line $y=x^D$ in the interval $x \in [0, 30]$.

In Fig. 2, we consider the difference between the charge $y=Q([0, x])$ of fractal distribution and the charge that is described by fractional continuous distribution in the interval $x \in [0, 30]$.

¹S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives Theory and Applications* (Gordon and Breach, New York, 1993).

²G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).

³G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).

⁴V. E. Tarasov and G. M. Zaslavsky, Physica A **354**, 249 (2005).

⁵V. E. Tarasov, Chaos **14**, 123 (2004).

⁶V. E. Tarasov, Phys. Rev. E **71**, 011102 (2005).

⁷V. E. Tarasov, J. Phys. Conf. Ser. **7**, 17 (2005).

⁸R. Nigmatullin, Phys. Status Solidi B **133**, 425 (1986).

⁹V. E. Tarasov, Phys. Lett. A **336**, 167 (2005).

¹⁰V. E. Tarasov, Phys. Lett. A **341**, 467 (2005).

¹¹V. E. Tarasov, Ann. Phys. **318**, 286–307 (2005).

¹²V. E. Tarasov, Chaos **15**, 023102 (2005).

¹³N. Engheta, IEEE Trans. Antennas Propag. **44**, 554 (1996).

¹⁴N. Engheta, IEEE Antennas Propag. Mag. **39**, 35 (1997).

¹⁵N. Engheta, Microwave Opt. Technol. Lett. **17**, 86 (1998).

¹⁶N. Engheta, in *Frontiers in Electromagnetics*, edited by D. H. Werner and R. Mittra (IEEE, New York, 2000), pp. 523–552.

¹⁷F. Y. Ren, J. R. Liang, X. T. Wang, and W. Y. Qiu, Chaos, Solitons Fractals **16**, 107 (2003).

¹⁸A. Le Mehaute, R. R. Nigmatullin, and L. Nivanen, *Fleches du temps et geometric fractale* (Hermes, Paris, 1998), Chap. 5.

¹⁹J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984).

²⁰R. M. Christensen, *Mechanics of Composite Materials* (Wiley, New York, 1979).

²¹J. Lutzen, "Liouville's differential calculus of arbitrary order and its electrodynamic origin," in proceeding of the 19th Nordic Congress Mathematicians (Icelandic Mathematical Society, Reykjavik, 1985), pp. 149–160.

²²J. Feder, *Fractals* (Plenum, New York, 1988).