interaction along a chain; and $t=[a(\alpha)]^{i,i+1}_{\alpha_i=A,B}=0.3$, which describes the interaction between the chains. Here, α_i is the species of the atom localized in the i-th chain for configuration α. For comparison we give the histogram of the density of states of a one-dimensional chain of 4,000 atoms calculated using the method of negative eigenvalues. Figure 2 illustrates the rapid convergence of the density of states with increasing number of chains and the vanishing of the singularities characteristic of the one-dimensional case. Thus, for $t/T = 0.3$ the density of states hardly changes already for $M \geqq 5$. With decreasing value of the parameter t convergence occurs even more rapidly, as the calculations showed.

As regards efficiency, the method proposed in this paper for analyzing the single-particle spectra of stochastic quasione-dimensional systems is not inferior to the method of negative eigenvalues, but at the same time it enables one to solve a much larger group of problems by virtue of the information contained in the averaged Green's functions.

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ULTRAVIOLET FINITENESS OF NONLINEAR TWO-DIMENSIONAL SIGMA MODELS ON AFFINE-METRIC MANIFOLDS

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The two-loop counterterms of a nonlinear two-dimensional boson sigma model whose target space is an arbitrary affine-metric manifold are calculated. Examples are given of nonflat manifolds that lead to ultraviolet-finite sigma models.

Study of nonlinear two-dimensional sigma models has recently become particularly interesting in connection with the development of string theory. The target manifolds of ultraviolet-finite sigma models determine the spaces of the compactified additional dimensions [1,2]. The condition of ultraviolet finiteness determines the equations of motion of the string modes [3–5].

The action of the bosonic sigma model has the form

$$I(\varphi)=\int d^2x\, G_{ij}(\varphi)\,\partial_\mu\varphi^i\,\partial^\mu\varphi^j, \tag{1}$$

where the integration is over a two-dimensional Minkowski space. The fields $\varphi^i(x)$ take values on some manifold M. It is generally assumed that M is a Riemannian manifold with metric-consistent connection [6,7]. In this case the condition of ultraviolet finiteness of the sigma model leads to the requirement of vanishing of the Riemann tensor. In other words, finiteness holds only for flat manifolds M.

In this paper we consider as space M an arbitrary affine-metric manifold for which consistency of the connection with the metric is not assumed. In this case the connection has the form

$$\Gamma^k{}_{ij}=\{^k{}_{ij}\}+D^k{}_{ij}, \tag{2}$$

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where $\left\{ {k \atop ij} \right\} = \frac{1}{2} G^{kp} (\partial_j G_{ip} + \partial_i G_{jp} - \partial_p G_{ij})$ is the Christoffel symbol,

$$D^k{}_{ij} = -{}^1/_2 G^{kp} (K_{ipj} + K_{jpi} - K_{ijp}) + 2Q_{(ij)}{}^k + Q^k{}_{ij} \tag{3}$$

is the connection defect, $K_{ijl} = \nabla_l G_{ij}$ is the nonmetricity tensor, $Q^k{}_{ij} = \Gamma^k{}_{[ij]}$ is the torsion tensor. For the Riemannian manifolds usually considered [6,7], the relations $K_{ijl} = 0$, $Q_{ijl} = 0$ hold.

The counterterms in the sigma model can be conveniently calculated by the background field method. In this method, the action is expanded in powers of the quantum field

$$\xi^i = \left. \frac{d\lambda^i(t)}{dt} \right|_{t=0},$$

where $\lambda^i(t)$ is the geodesic determined by the equation

$$\frac{d^2\lambda^k}{dt^2} + \Gamma^k{}_{ij} \frac{d\lambda^i}{dt} \frac{d\lambda^j}{dt} = 0.$$

Note that the symmetric part of the connection (2) occurs in the equation for the geodesic in the considered case. The covariant expansion of the action (1) in powers of the quantum field differs from the corresponding expansion for Riemannian manifolds [7,8] by the presence of the additional terms

$$\begin{gathered}
I^{(2)} = \int d^2x \, \{ G_{ij} \nabla_\mu \xi^i \nabla^\mu \xi^j + 2G_{ij;k} \xi^k \nabla_\mu \xi^i \partial^\mu \varphi^j + (\overset{s}{\mathcal{R}}_{iklj} + {}^1/_2 G_{ij;k;l}) \xi^k \xi^l \partial_\mu \varphi^i \partial^\mu \varphi^j \}, \\
I^{(3)} = \int d^2x \, \{ G_{ij;k} \xi^k \nabla_\mu \xi^i \nabla^\mu \xi^j + [{}^4/_3 \overset{s}{\mathcal{R}}_{\underline{iklj}} + G_{ij;k;l}] \xi^k \xi^l \nabla_\mu \xi^i \partial^\mu \varphi^j + \\
[{}^2/_3 G_{pj;k} \overset{s}{\mathcal{R}}{}^p{}_{lmi} + {}^1/_3 \overset{s}{\mathcal{R}}_{iklj;m} + {}^1/_6 G_{ij;k;l;m}] \xi^k \xi^l \xi^m \partial_\mu \varphi^i \partial^\mu \varphi^j \}, \\
I^{(4)} = \int d^2x \, \{ [{}^1/_3 \overset{s}{\mathcal{R}}_{iklj} + {}^1/_2 G_{ij;k;l}] \xi^k \xi^l \nabla_\mu \xi^i \nabla^\mu \xi^j + \\
[{}^5/_6 G_{\underline{pi;k} \overset{s}{\mathcal{R}}{}^p{}_{lmj}} + {}^1/_2 \overset{s}{\mathcal{R}}_{\underline{iklj};m} + {}^1/_3 G_{ij;k;l;m}] \xi^k \xi^l \xi^m \nabla_\mu \xi^i \partial^\mu \varphi^j + \\
[{}^1/_3 \overset{s}{\mathcal{R}}_{pklj} \overset{s}{\mathcal{R}}{}^p{}_{mni} + {}^1/_{12} \overset{s}{\mathcal{R}}_{iklj;m;n} + {}^1/_{24} G_{ij;k;l;m;n} + \\
{}^5/_{12} G_{pj;k;l} \overset{s}{\mathcal{R}}{}^p{}_{nmi} + {}^1/_6 G_{pj;k} \overset{s}{\mathcal{R}}{}^p{}_{lmi;n}] \xi^k \xi^l \xi^m \xi^n \partial_\mu \varphi^i \partial^\mu \varphi^j \}.
\end{gathered} \tag{4}$$

We have here introduced the notation

$$\begin{gathered}
\overset{s}{\mathcal{R}}{}^i{}_{jkl} = \mathcal{R}^i{}_{jkl} + 2Q^i{}_{j[k;l]} + 2Q^n{}_{j[k|} Q^i{}_{n|l]}, \qquad \mathcal{R}^i{}_{jkl} = 2\partial_{[k|} \Gamma^i{}_{j|l]} + 2\Gamma^i{}_{n[k|} \Gamma^n{}_{j|l]}, \\
A_{i;j} = \overset{s}{\nabla}_j A_i = \nabla_j A_i + Q^k{}_{ji} A_k, \qquad \nabla_j A_i = \partial_j A_i - \Gamma^k{}_{ji} A_k, \\
\nabla_\mu \xi^i = \partial_\mu \xi^i + \Gamma^i{}_{(pj)} \xi^j \partial_\mu \varphi^p, \qquad A_{\underline{ijk}} = A_{(i|j|k)} = {}^1/_2 (A_{ijk} + A_{kji}), \qquad A_{[i|j|k]} = {}^1/_2 (A_{ijk} - A_{kji}).
\end{gathered}$$

In each order, the obtained counterterms have the structure of Eq. (1) and reduce to a renormalization of the metric tensor G_{ij} [6].

The divergent single-loop counterterm is

$$T^{(1,1)}{}_{ij} = \frac{1}{4\pi\varepsilon} \left(\overset{s}{\mathcal{R}}_{(ij)} + \frac{1}{2} G_{ij;a;a} - 2G_{i[a;b]} G_{j[a;b]} \right), \tag{5}$$

where $2\mathcal{R}_{ab} = \mathcal{R}_{acdb} G^{cd}$. In calculating the counterterms, we use dimensional regularization, $2 \to n = 2 - 2\varepsilon$ and introduce an auxiliary mass term to eliminate the infrared divergences.

The divergent two-loop counterterm (including the contribution obtained from the expansion of the expression (5) to terms quadratic in the quantum field) contains terms proportional to ε^{-2} and ε^{-1}. The term proportional to ε^{-2} can be obtained from (5) by means of pole equations [7]. Therefore, we write down the answer only for the coefficient of a simple pole:

$$T^{(2,1)}{}_{ij} = \frac{1}{16\pi^2\varepsilon} \left\{ \frac{16}{9} \overset{s}{\mathcal{R}}_{i\underline{(a[b)c]}} \overset{s}{\mathcal{R}}_{j\underline{(a[b)c]}} + \frac{1}{3} \overset{s}{\mathcal{R}}_{i(ab)j} \overset{s}{\mathcal{R}}_{(ab)} - \right.$$

$$
\begin{aligned}
&{}^{1}/_{3}\,\overset{s}{\mathcal{R}}_{i(ab)j}\overset{s}{\mathcal{R}}_{c(ab)c} - {}^{1}/_{3}\,G_{i(a;\,b);\,j}\,(5\overset{s}{\mathcal{R}}_{(ab)} + \overset{s}{\mathcal{R}}_{c(ab)c}) - \\
&{}^{1}/_{6}\,G_{ij;\,(a;\,b)}\,(\overset{s}{\mathcal{R}}_{c(ab)c} - 2\overset{s}{\mathcal{R}}_{(ab)}) - {}^{1}/_{2}\,\overset{s}{\mathcal{R}}_{i(ab)j}\,(G_{cc;\,(a;\,b)} + G_{ab;\,c;\,c}) + \\
&{}^{8}/_{3}\,G_{i[c;\,b];\,a}\overset{s}{\mathcal{R}}_{\underbrace{j(a[b)c]}} - 2G_{ac;\,c;\,b}\overset{s}{\mathcal{R}}_{i(ab)j} + {}^{32}/_{3}\,G_{i[p;\,q]}G_{p(b;\,a)}\overset{s}{\mathcal{R}}_{\underbrace{j(a[b)q]}} - {}^{5}/_{2}\,G_{j[a;\,b]}G_{pb;\,(c}\overset{s}{\mathcal{R}}^{p}{}_{ca)i} + \\
&{}^{1}/_{2}\,\overset{s}{\mathcal{R}}_{i(ab)j}\,(8G_{p[q;\,a]}G_{p[q;\,b]} - G_{pq;\,a}G_{pq;\,b} - 4G_{a[p;\,q]}G_{b[p;\,q]}) + \\
&{}^{4}/_{3}\,\overset{s}{\mathcal{R}}_{\underbrace{c(ab)d}}G_{i[a;\,c]}G_{j[b;\,d]} + {}^{8}/_{9}\,G_{i[d;\,a]}G_{j[d;\,b]}\,(\overset{s}{\mathcal{R}}_{c(ab)c} + {}^{3}/_{4}\,\overset{s}{\mathcal{R}}_{(ab)}) - \\
&{}^{1}/_{2}\,G_{j[a;\,b]}\,(3\overset{s}{\mathcal{R}}_{b(cc|i;\,|a)} + 4\overset{s}{\mathcal{R}}_{(ai);\,b}) - {}^{8}/_{3}\,G_{a[c;\,b]}\overset{s}{\mathcal{R}}_{i(ab)c;\,j} + \\
&G_{i[a;\,b]}\,(G_{jb;\,c;\,c;\,a} - G_{jb;\,(c;\,c;\,a)}) - 2G_{a[c;\,b]}G_{ic;\,(a;\,b);\,j} + \\
&G_{ca;\,b;\,i}G_{ab;\,c;\,j} - {}^{1}/_{2}\,G_{i(a;\,b);\,j}\,(G_{cc;\,(a;\,b)} + 4G_{ac;\,c;\,b}) - G_{ij;\,(a;\,b)}G_{ac;\,c;\,b} + 8G_{i[p;\,q]}G_{p(b;\,a)}G_{j[q;\,(a];\,b)} + \\
&2G_{i[a;\,c];\,j}G_{cp;\,q}G_{q[p,\,a]} + {}^{1}/_{4}\,G_{ij;\,(p;\,q)}\,(8G_{a[b;\,p]}G_{a[b;\,q]} - G_{ab;\,p}G_{ab;\,q} - \\
&4G_{p[a;\,b]}G_{q[a;\,b]}) + 2G_{cd;\,(a;\,b)}G_{i[a;\,c]}G_{j[b;\,d]} + {}^{1}/_{3}\,G_{i[p;\,a]}G_{j[p;\,b]} \times \\
&(5G_{ab;\,c;\,c} + 4G_{cc;\,a;\,b}) + 4G_{i(a;\,b);\,j}G_{a[p;\,q]}G_{b[p;\,q]} + \\
&8G_{i[a;\,b]}G_{j[c;\,d]}G_{a[c;\,d];\,b} + {}^{16}/_{3}\,G_{i[a;\,b]}G_{j[a;\,c]}G_{(b|p;\,p|;\,c)} + \\
&{}^{4}/_{3}\,G_{i[a;\,b]}G_{j[a;\,c]}G_{b[p;\,q]}G_{c[p;\,q]} - {}^{5}/_{3}\,G_{i[a;\,b]}G_{j[c;\,d]}\,(G_{ac;\,p}G_{bd;\,p} - \\
&4G_{pa;\,c}G_{bd;\,p} + 6G_{pa;\,c}G_{pd;\,b}) + {}^{20}/_{9}\,G_{i[a;\,b]}G_{j[a;\,c]}\,(G_{pc;\,q}G_{qb;\,p} - 4G_{pq;\,b}G_{p(q;\,c)} + 2G_{pb;\,q}G_{pc;\,q})\}.
\end{aligned}
\tag{6}
$$

Repeated subscripts denote summation with the tensor $\frac{1}{2}G^{ab}$, for example,

$$B_{aa;\,c}A_c \equiv {}^{1}/_{4}G^{ab}G^{cd}B_{ab;\,c}A_d, \quad \mathcal{R}_{ab} \equiv {}^{1}/_{2}G^{cd}\mathcal{R}_{acdb}.$$

It is readily shown that on the transition to a Riemannian manifold Eqs. (5) and (6) lead to the well-known expressions [6,7].

We give examples of nonflat manifolds for which the single-loop and two-loop counter-terms (5) and (6) vanish. In particular, this occurs if we have fulfillment of the conditions

$$K_{ijl} = N_{ijl} - 2Q_{(ij)l}, \quad N_{ijl} = K_{(ijl)}, \quad N_{ijl;\,k} = 0, \quad \overset{s}{\mathcal{R}}_{iklj} = 0 \tag{7}$$

or

$$Q_{ijl} = 0, \quad K_{ijl} = K_{(ijl)}, \quad \mathcal{R}_{iklj} = -{}^{1}/_{2}\nabla_{(k}K_{l)ij}. \tag{8}$$

In these cases the Riemann tensor $R^{k}{}_{mln} = \partial_l \{^{k}{}_{mn}\} - \partial_n \{^{k}{}_{ml}\} + \{^{k}{}_{pl}\}\{^{p}{}_{mn}\} - \{^{k}{}_{pn}\}\{^{p}{}_{ml}\}$ is, respectively,

$$R_{iklj} = {}^{1}/_{2}N_{mi[l|}N^{m}{}_{k|j]} + 2Q_{(mi)[l}N^{m}{}_{|k|j]}, \qquad R_{iklj} = {}^{1}/_{2}\,K^{m}{}_{i[l}K_{j]mk} - {}^{1}/_{2}\,\nabla_{(k}K_{l)ij}.$$

Just as finiteness of the sigma model with Wess–Zumino term on parallelizable manifolds was proved [9], we should be able to show that in each order of perturbation theory ultraviolet finiteness of the sigma model on the affine-metric manifold is ensured by the condition

$$\overset{s}{\mathcal{R}}_{iklj} + {}^{1}/_{2}G_{ij;\,k;\,l} - 2G_{i[k;\,a]}G_{j[l;\,a]} = 0,$$

special cases of which are provided by the conditions (7) and (8).

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STRING OPERATOR FORMALISM AND FUNCTIONAL INTEGRAL IN THE HOLOMORPHIC REPRESENTATION

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The connection between a functional integral over open Riemann surfaces [1] and the operator formalism on closed Riemann surfaces [2] is discussed. The states in the operator formalism are a holomorphic representation of the functional integral.

1. Several recent papers [1,2] have been devoted to the calculation of functional integrals I_Σ over Riemann surfaces Σ with boundary Γ. The integrals I_Σ are important in string theory, since they satisfy the so-called sewing algebra: if the surface Σ is obtained from the surfaces Σ_1 and Σ_2 by identifying some components of the boundary of the surface Σ_1 with the same number of components of the boundary of the surface Σ_2, then

$$I_\Sigma = I_{\Sigma_1} \overset{*}{\gamma} I_{\Sigma_2},$$

where γ is the common part of the boundary of the surfaces Σ_1 and Σ_2, and $\overset{*}{\gamma}$ is some multiplication, which will be described below. Using this "sewing," we can construct I_Σ for a surface of arbitrarily high genus from simple blocks, for example, from functional integrals over "trousers" (spheres with three deleted disks).

In the coordinate approach [1] for scalars with action $S=\frac{1}{\pi}\int \partial\Phi\bar{\partial}\Phi\, d^2\xi$ I_Σ is a functional of the values of the field on the boundary and is equal to

$$I_\Sigma(\Phi_\Gamma) = (\det_0 \Delta_\Sigma)^{-1/2} \exp(-S(\Phi_h(\Phi_\Gamma))), \tag{1}$$

where $\Phi_h(\Phi_\Gamma)$ is a harmonic function on Σ, equal to Φ_Γ on the boundary, and $\det_0 \Delta_\Sigma$ is the determinant on functions on Σ with zero-value boundary conditions. In this approach, "sewing" along the common boundary γ means integration over the fields on γ:

$$I_{\Sigma_1} \overset{*}{\gamma} I_{\Sigma_2} = \int D\Phi_\gamma I_{\Sigma_1} \cdot I_{\Sigma_2}.$$

The operator approach [2] considers quantization near boundaries, which leads to a certain Heisenberg algebra. In this approach, I_Σ is regarded as a state in the representation space of this algebra that can be obtained from the vacuum by a certain Bogolyubov transformation, the "sewing" being specified by the scalar product in the Hilbert space corresponding to the common boundary.

In this paper we show that for scalars the operator formalism is the holomorphic representation for the functional integral.

2. For simplicity we consider the case of a connected boundary. The generalization to the case of a disconnected boundary is trivial.

It follows from (1) that $S(\Phi_\Gamma)$ determines $I_\Sigma(\Phi_\Gamma)$ up to a constant factor $(\det_0 \Delta_\Sigma)^{-\frac{1}{2}}$. Therefore, we first calculate $S(\Phi_\Gamma)$, and we then find $\det_0 \Delta_\Sigma$ from the sewing algebra.

Let z be a coordinate near Γ such that $|z| = 1$ on the boundary and $|z| > 1$ for other points of Σ near the boundary. The field Φ_Γ on the boundary can be specified by means of coefficients Φ_n in the following Fourier series:

$$\Phi_\Gamma = \Phi_0 + \sum_{n\neq 0} \frac{\Phi_n}{\sqrt{|n|}} e^{in\varphi}, \quad \varphi = \arg z.$$

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