Theoretical and Mathematical Physics, Vol. 100, No. 3, 1994

QUANTUM DISSIPATIVE SYSTEMS. I. CANONICAL QUANTIZATION AND QUANTUM LIOUVILLE EQUATION

V. E. Tarasov

Sedov's variational principle, which is a generalization of the principle of least action to dissipative processes, is used to generalize canonical quantization and the von Neumann equations to dissipative systems. The example of a harmonic oscillator with friction is considered.

1. INTRODUCTION

Newtonian vector mechanics describes the motion of mechanical systems under the action of forces applied to them. The Newtonian approach does not restrict the nature of the acting forces, which are usually divided into potential and dissipative forces. The Lagrange—Hamilton variational mechanics describes the motion of mechanical systems under the influence of only potential forces [1, 2]. Dissipative forces are outside the field of applicability of the variational principles of analytical mechanics [1, 3—8]. It is on account of this restriction that statistical mechanics does not describe irreversible and dissipative processes. This is due to the fact that in the framework of Hamiltonian dynamics there does not exist a function of the coordinates, momenta, and time possessing the properties of a Lyapunov function (Poincaré—Misra theorem [9—12]). To describe dissipative and irreversible processes, it is necessary to introduce into statistical mechanics additional postulates (for example, the principle of correlation weakening and the hypothesis of a hierarchy of relaxation times as proposed by Bogolyubov [13, 14]). Therefore, these processes are considered in the framework of physical kinetics [15, 16]. It is well known that for the construction of a quantum theory the point of departure is the Hamiltonian form of the classical mechanics [17, 18]. Therefore, quantum mechanics describes only conservative physical systems. Irreversible and dissipative quantum dynamics is outside the framework of quantum mechanics and quantum statistics. The quantum description of dissipative and irreversible processes is the aim of quantum kinetics, which is quantum statistics augmented with additional physical postulates [15].

The study of dissipative systems in quantum theory is of great theoretical interest and is important for practical applications. We note some directions that indicate the importance of the quantum description of dissipative systems. First, measurements made on a system or ensemble cannot be described by means of the linear laws of quantum mechanics. The laws of quantum mechanics make it possible to obtain only probabilistic correlations between the results of several successive measurements [19]. In the framework of quantum mechanics, it is not possible to obtain a complete description of measurement [19]. This necessitates truncation of the infinite sequence of measuring instruments and the introduction of two fundamentally different types of changes of the state of a system or an ensemble with the passage of time [20]. In connection with this asymmetry of the theory due to the duality of the description, it has often been argued that it is necessary to generalize the quantum-mechanical equations of motion to irreversible and dissipative processes in which entropy changes [19, 21, 12]. At the same time, it is important that the change of the entropy be objective and not depend on the rather subjective concept of coarse graining [21]. Second, the process of formation and evaporation of a black hole cannot be described by a S matrix, since the evolution takes place with a change of the entropy [22]. As a black hole evolves, a pure quantum state decays into a mixed state [23]. We note that the concept of entropy in the physics of black holes is an objective intrinsic property of the system and is not associated with a coarse-grained description [24, 25]. In addition, the interaction of microscopic systems (elementary particles) with a black hole leads to the need for a nonunitary generalization of von Neumann's equation [26]. Third, in the physics of elementary particles dissipative systems may play a more important role than has hitherto been accorded to them. The reasons for this are the following. Decay of a pure quantum state to a mixed state can occur at the level of elementary particles on account of quantum fluctuations of the metric that are virtual black holes [23]. Departure from an exponential law [27, 28] in quantum mechanics can lead to serious problems concerning the meaning of particle identity [12]. Although it has

Institute of Nuclear Physics, State University, Moscow. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 100, No.3, pp. 402-417, September, 1994. Original article submitted May 25, 1993.

 0040-5779/94/1003-1100$12.50

proved possible to include the violation of T invariance in the decays of K^0 mesons in the standard model of electroweak interactions, the fundamental physical principles of this effect may be associated with dissipative models. In addition, dissipation effects may play a certain role in string theories [30]. Fourth, the phenomenological description of nuclear friction, which is manifested in the deep inelastic scattering of heavy ions and in fission processes, makes it necessary to consider quantum dissipative models [29]. It is well known that dissipative systems play an important role in the physics of continuous media [4, 6, 7, 8].

We mention some of the approaches to the solution of the problems of the quantum description of dissipative systems. One of the approaches is associated with canonical quantization of dissipative systems [31, 32]. It is well known that the equations of motion are Euler—Lagrange equations constructed from a Lagrangian or are equivalent to such equations if Helmholtz's conditions are satisfied [35, 36]. There has been proposed a theory of general multipliers that make it possible, using Helmholtz's conditions, to construct a Lagrangian formulation for a large class of equations of motion [37] that usually cannot be included in Lagrangian or Hamiltonian mechanics. The canonical quantization of systems defined by Lagrangians of the above types is either impossible or arbitrary [38, 37]. That is, among all admissible (s-equivalent) Lagrange functions for a given equation, we do not know which of them to choose for the quantization procedure. Although the existence of a classical Hamiltonian is necessary for canonical quantization [18, 17], this is not sufficient for the construction of a quantization in a satisfactory form [39]. The canonical quantization of systems whose Hamiltonians are not canonically related to the energy of the given system is arbitrary, and the results contradict the physical interpretation. It was shown [35] that although there exists a class of inequivalent (s-equivalent) Lagrangians, quantization of the systems described by them is impossible, since the Lagrangian must be not only related to the equation of motion but also generate a Hamiltonian that is canonically conjugate to the physical energy of the system. However, this condition can be satisfied only for conservative systems, ruling out the possibility of canonical quantization of dissipative systems. A similar conclusion — incompatibility of dissipative systems with the canonical commutation relations — was obtained in [33, 34]. We note that in the proof of incompatibility in [34], total derivatives with respect to the time of the commutation relations for the coordinates and the momenta were considered, and the Jacobi identity and dissipative equations of motion for a Heisenberg operator were used. The important connection between Helmholtz's conditions and the quantum commutation relations was revealed in [40] on the basis of classical equations of motion and the most general quantum conditions, namely, commutation of the coordinate operators. By considering the total derivative of these commutation relations, it was shown that the commutator of the coordinate and velocity operators forms a symmetric tensor operator. The classical analog of this tensor operator is a matrix whose inverse satisfies Helmholtz's conditions. Using the Jacobi identities for the coordinate and velocity operators, it was shown that the general quantum conditions presuppose equivalence of the equations of motion with the Euler—Lagrange equations.

Other attempts to solve the problem of a quantum dynamical description of dissipative systems are associated with the generalization of quantum statistics and von Neumann's equations. An important property of dissipative and irreversible processes is the increase of entropy. However, the quantum-mechanical evolution equations for the statistical operator (the operator of the density matrix), which are called the von Neumann equations, keep the entropy unchanged. Generalizations of the von Neumann equation to dissipative and irreversible processes are usually obtained by adding a superoperator, which acts on the statistical operator and describes the dissipative, irreversible part of the evolution of the system. Linear generalizations of the von Neumann equation are associated with a master equation (Pauli equation) obtained in the framework of quantum kinetics [34] or with a quantum dynamical semigroup [41, 42, 43]. It has been shown [44] that the generators of quantum dynamical semigroups violate the algebraic structure. It is also known that the total time derivative for the operators of dissipative systems does not satisfy Leibniz's rule, i.e., is not an operator of differentiation in the strict sense of the work and is called a dissipative operator [45]. There has also been consideration of nonlinear generalizations of von Neumann's equations [46—50] corresponding to the nonlinear Schrödinger equations proposed earlier in [51—53] for the description of dissipative systems. We note that the proposed generalizations of the von Neumann equations are obtained heuristically or by introducing postulates additional to quantum statistics. The requirements that have been proposed in order to determine the superoperator uniquely are themselves not unique, and therefore one must come to terms with the problems that arise from this arbitrariness, or construct a description in the framework of quantum kinetics. In addition, the proposed generalizations of the von Neumann equations are not associated with the classical Liouville equation for dissipative processes that was already proposed by Liouville [54] and considered in [55—57]. We note that Lie derivatives can be used to obtain these equations, and it is easy to obtain an expression for the classical rates of entropy production [57]. Thus, a quantum description of dissipative systems using generalizations of the von Neumann equations and not associated with the classical dissipative Liouville equations is arbitrary or possible only in the framework of quantum kinetics.

In connection with the difficulties of the canonical quantization of dissipative systems, noncanonical quantization schemes have been proposed [58], namely, operator extensions of the Hamilton-like and Birkhoff-like generalizations of classical Hamiltonian mechanics. It was shown [58, 59] that the equations of the evolution in time for dissipative systems not only destroy

the structure of the Lie algebra but also do not determine in general any algebra, since the distributive law is violated. To preserve the algebraic structure, it was proposed to construct the quantum dynamics of dissipative systems in the framework of nonassociative algebra. This construction was realized in terms of noncanonical quantization at the level of nonassociative Lie-admissible and Lie-isotopic enveloping algebras [58—60] and in the framework of a nonassociative flexible Lie algebra [62]. A generalization of the von Neumann equations was obtained in the framework of the nonassociative Lie-admissible approach [63]. We note that the generalized variations used in the given noncanonical approach [58, 59, 61] to consider dissipative processes in the sphere of holonomic variational principles are associated with generalized multipliers [38, 37] and, therefore, with the problems of the arbitrariness in the choice of the holonomic function that generates the equations of motion. In addition, all operators, including the coordinate and momentum operators, which satisfy noncanonical commutation relations become nonassociative, and this complicates the description.

Sedov [3—7] proposed a variational principle that is a generalization of the principle of least action to dissipative and irreversible processes. To include dissipative processes in the sphere of applicability of variational principles, he proposed that one should consider not only holonomic but also nonholonomic objects. The proposed variational principle was used in [8, 64—68] to construct phenomenological models of continuous media with irresversibility and dissipation. A special case of Sedov's variational principle is the Helmholtz—Planck variational principle [35, 69], which is mentioned in [2, 61, 70, 36].

Classical mechanics in phase space was considered on the basis of Sedov's variational principle [71]. The proposed form of dissipative Hamiltonian mechanics was used in [71] to generalize canonical quantization and the von Neumann equations to dissipative processes. The obtained operator algebra is a natural extension of the canonical commutation relations through the introduction of operators of nonholonomic quantities in addition to the usual associative operators of holonomic functions of the coordinates, momenta, and time. These commutation relations are obtained by applying the standard procedure to the classical Poisson brackets. If all the proposed commutation relations are to hold, the operators of a nonholonomic quantity must be a nonassociative non-Lie (not satisfying the Jacobi identity) operator. At the same time, it is sufficient to require left—right nonassociativity but retain left and right associativity. As a result, the action of a total time derivative on a product of operators does not satisfy Leibniz's rule, which is deformed by the appearance of the associator of the operator of the nonholonomic quantity. A generalization was proposed for the von Neumann equation, which is the quantum analog of the dissipative Liouville equation. Dissipative analogs of the Heisenberg and Schrödinger equations were obtained [71], and the Feynman representation of the Green's function for the generalized Schrödinger equation was considered. The proposed dissipative quantum scheme makes it possible to formulate an approach to the construction of quantum dissipative field theory and the quantization of phenomenological models of continuous media [8, 64—68]. As an example of dissipative quantum field theory, the sigma-model approach [72—75] to quantum string theory [76] was considered [71, 77].

In this paper, we discuss in detail the construction of the Hamiltonian and the quantum description of dissipative systems proposed in [71].

2. SEDOV'S VARIATIONAL PRINCIPLE

The equations of motion of mechanical systems in an n-dimensional configuration space are written in the form

$$D_i T(q,u,t) + Q_i = 0, \tag{1}$$

where T is the kinetic energy, which can be represented in the form

$$T(q,u,t) = \frac{1}{2}a_{ij}(q,t)u^i u^j + a_i(q,t)u^i + a_0(q,t), \quad D_i \equiv \frac{d}{dt}\frac{\partial}{\partial u^i} - \frac{\partial}{\partial q^i}, \tag{2}$$

where $i, j=1,\ldots,n$; $u^i \equiv dq^i/dt$; and $Q_i = Q_i(q, u, t)$ is the sum of the external forces. In the general case, Q_i is the sum of potential, Q_i^p, and dissipative Q_i^p, forces. A force is called a potential force is there exists for it a function $V=V(q, u, t)$: $D_i V = -Q_i^p$. Forces Q_i^d are said to be dissipative if they cannot be represented in such a form. Then the Euler—Lagrange equations take the form

$$D_i L + Q_i^d = 0, \tag{3}$$

where $L=L(q, u, t) \equiv T-V$ is the Lagrangian. In the dissipative case ($Q^d \not\equiv 0$), Eq. (3) cannot be obtained from the principle of least action [1,2]:

$$\delta S(q) \equiv \delta \int dt\, L(q,u,t) = 0. \tag{4}$$

As basic variational principle for the description of dissipative processes, we propose to consider Sedov's variational principle

[8], which is a generalization of the principle of least action and has the form

$$\delta S(q) + \delta \tilde{W}(q) = 0, \tag{5}$$

where $S(q)$ is a holonomic functional, called the action, and $\tilde{W}(q)$ is a nonholonomic functional. The nonholonomic functional is determined by a nonholonomic equation. We assume that the variation of the nonholonomic functional is linear in the variations δq^i and δu^i:

$$\delta \tilde{W} = \delta \int dt\, w(q,u) = \int dt\, (w_i^1(q,v)\delta q^i + w_i^2(q,u)\delta u^i), \tag{6}$$

where w_i^1 and w_i^2 are vector functions in the configuration space. We now turn to consideration of the Hamiltonian approach to the classical mechanics of dissipative systems.

3. HAMILTONIAN DISSIPATIVE MECHANICS

From Sedov's variational principle (5) with nonholonomic functional (6) there follow dissipative equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L(q,u)}{\partial u^i} + w_i^2(q,u)\right) = \frac{\partial L(q,u)}{\partial q^i} + w_i^1(q,u)\ , \qquad \frac{dq^i}{dt} = u^i. \tag{7}$$

We define the canonical momentum by the equation

$$p_i \equiv \frac{\partial L(q,u)}{\partial u^i} + w_i^2(q,u), \tag{8}$$

and represent this relation in the form $u^i = v^i(q, p)$, assuming for simplicity that Eq. (8) is solvable. The Hamiltonian can be determined in the standard form

$$h(q,p) = p_i v^i(q,p) - L(q, v(q,p)). \tag{9}$$

Considering the variation of the Hamiltonian, we readily obtain dissipative Hamilton equations

$$\frac{dq^i}{dt} = \frac{\delta(h-w)}{\delta p_i}, \qquad \frac{dp_i}{dt} = -\frac{\delta(h-w)}{\delta q^i}\ , \tag{10}$$

where

$$\delta w(q,p) = \delta w(q, v(q,p)) = w_i^q \delta q^i + w_p^i \delta p_i. \tag{11}$$

We assume that the coordinates z^k and w, t, where $k=1,\ldots,2n$, $z^i=q^i$, $z^{n+i}=p_i$ $(i=1,\ldots,n)$, of the $(2n+2)$-dimensional extended phase space are related by the equation

$$\delta w - a_k(z,t)\delta z^k = 0, \tag{12}$$

where a_k $(k=1,\ldots,2n)$ is vector function in the phase space. We shall call the dependence of w on the coordinates q and the momenta p a holonomic—nonholonomic function and denote it by $w=w(z)\in\Phi$. If the vector function satisfies the condition

$$\frac{\partial a_k(z)}{\partial z^l} = \frac{\partial a^l(z)}{\partial z^k}, \tag{13}$$

where k, $l=1,\ldots,2n$, then w is a holonomic function $(w\in F)$. If this vector function does not satisfy the condition (13), then we shall call the object $w(z)$ a nonholonomic function, or a Sedovian $(w\in\tilde{F})$. We define the variational Poisson brackets for $\forall a$, $b\in\Phi$ in the form

$$[f,g] \equiv \frac{\delta f}{\delta q^i}\frac{\delta g}{\delta p_i} - \frac{\delta f}{\delta p_i}\frac{\delta g}{\delta q^i} \tag{14}$$

and give the main properties of these brackets:

1) antisymmetry:

$$\forall f, g \in \Phi \quad [f,g] = -[g,f] \in F;$$

2) Jacobi identity:

$$\forall f, g, s \in F \quad J[f, g, s] = 0;$$

3) non-Lie property:

$$\forall f, g, s \in \Phi : \ f \vee g \vee s \in \tilde{F} \quad J[f, g, s] \not\equiv 0;$$

4) Leibniz's rule:

$$\forall f, g \in \Phi \ \ (\partial[f, g])/(\partial t) = [\partial f/\partial t, g] + [f, \partial g/\partial t];$$

5) the distributive property:

$$\forall f, g, s \in \Phi \ \ [\alpha f + \beta g, s] = \alpha[f, s] + \beta[g, s] \ ,$$

where

$$J[f, g, s] \equiv [f, [g, s]] + [g, [s. f]] + [s, [f, g]] \ ,$$

and α and β are real numbers. It can be seen that these properties of the variational Poisson brackets for holonomic functions are identical to the properties of ordinary Poisson brackets [1, 2]. We consider the characteristic properties of the main physical quantities:

$$\begin{array}{ll}
1) & [p_i, p_j] = [q^i, q^j] = 0 \ , \quad [q^i, p_j] = \delta^i_j, \\
2) & [w, p_i] = w^q_i \ , \quad [w, q^i] = -w^i_p, \\
3) & [[w, p_i], p_j] \not\equiv [[w, p_j], p_i] \quad \text{or} \quad J[p_i, w, p_j] = \Omega_{ij} \not\equiv 0, \\
4) & [[w, q^i], q^j] \not\equiv [[w, q^j], q^i] \quad \text{or} \quad J[q^i, w, q^j] = \Omega^{ij} \not\equiv 0, \\
5) & [[w, q^i], p_j] \not\equiv [[w, p_j], q^i] \quad \text{or} \quad J[q^i, w, p_j] = \Omega^i_j \not\equiv 0,
\end{array}$$

where, for example, the last omega-tensor can be determined from the equation

$$\Omega^i_j \equiv \frac{\partial w^q_j}{\partial p_i} - \frac{\partial w^i_p}{\partial q^j} = \frac{\delta^2 w}{\delta p_i \delta q^j} - \frac{\delta^2 w}{\delta q^j \partial p_i}. \tag{15}$$

The tensors Ω^{kl} characterize the deviation from the integrability condition (13) for Eq. (12), and by Stokes's theorem

$$\oint_{\partial M} \delta w = \int_M \Omega^{kl} \, dz^k \wedge dz^l \not\equiv 0. \tag{16}$$

We mention that some of the properties (3)—(5) need not be satisfied, but at least one of them must hold if we consider dissipative processes. Taking into account the definition of the variational Poisson brackets, we can represent the dissipative analog of the Hamiltonian equations of motion (10) in the form

$$\frac{dq^i}{dt} = [q^i, h - w], \quad \frac{dp_i}{dt} = [p_i, h - w]. \tag{17}$$

The total time derivative of a physical variable $A=A(q, p, t)\in F$ can be written in the form

$$\frac{dA(q, p, t)}{dt} = \frac{\partial A(q, p, t)}{\partial t} + [A, h - w]. \tag{18}$$

The equations of motion (17) can be obtained from Eq. (18) as a special case. Note that any term that is simultaneously added to the Hamiltonian h and to the Sedovian w does not change the equations of motion (17)—(18). This arbitrariness in the definition of the Hamiltonian can be readily eliminated by requiring that the Hamiltonian be canonically conjugate to the physical energy of the system [39]. It is readily seen that the total time derivative of the classical Poisson brackets does not satisfy Leibniz's rules

$$\frac{d}{dt}[f, g] = \left[\frac{d}{dt} f, g\right] + \left[f, \frac{d}{dt} g\right] + J[f, w, g]. \tag{19}$$

We consider a solution of Eq. (17) in the form

$$q^i = q^i(q_0, p_0, t) \ , \quad p_i = p_i(q_0, p_0, t). \tag{20}$$

We assume that points lying in the volume $V_0 = \int \delta q_0 \delta p_0$ of the phase space are initial points at the time $t=t_0$. Then Eq. (20) transforms the volume V_0 into the volume $V = \int \delta q \delta p = \int I \delta q_0 \delta p_0$, where $I=\partial(q, p)/\partial(q_0, p_0)=[q^i, p_i]_0$ is the Jacobian. It is

readily verified that [54]

$$\frac{dV}{dt} = \int \delta q \delta p \, \Omega, \tag{21}$$

where

$$\Omega = \sum_{i=1}^{n} \Omega_i^i = \sum_{i=1}^{n} J[q^i, w, p_i].$$

In accordance with the fundamental postulate of statistical mechanics [16], the state of a physical system at a certain time is determined by the probability density distribution function $\rho(q, p, t)$ which satisfies the normalization condition

$$\int dq dp \, \rho(q, p, t) = 1, \tag{22}$$

and the mean value of some physical variable $A(q, p, t)$ is determined in the form

$$\langle A(t) \rangle_{\rho(t)} = \int dq dp \, \rho(q, p, t) \, A(q, p, t). \tag{23}$$

Using for Eq. (22) the expression (21), we obtain in the case of a flat phase space the dissipative Liouville equation [54, 56, 57]

$$\frac{d\rho}{dt} = -\Omega\rho \quad \text{or} \quad i\frac{\partial \rho}{\partial t} = \hat{L}\rho, \tag{24}$$

where

$$\hat{L} = i\left(\frac{\delta(h-w)}{\delta q^k}\frac{\partial}{\partial p_k} - \frac{\delta(h-w)}{\delta p_k}\frac{\partial}{\partial q^k} - \Omega\right) \tag{25}$$

is called the Liouville operator [12, 16]. In addition to the Poincaré—Misra theorem [9—12], one can prove the existence in dissipative Hamiltonian mechanics of a function of the coordinates and momenta that is a Lyapunov function. To show this, we define the function $\eta(q, p, t) \equiv -\rho(q, p, t)$ and set $\Omega > 0$. Equation (24) shows that $d\eta/dt = \Omega$, and the function η satisfies the relation $d\eta/dt > 0$. It is convenient to define the entropy of the distribution density [12, 16] as follows:

$$s \equiv \langle \eta \rangle = -\int \delta q \delta p \, \rho(q, p, t) \ln \rho(q, p, t). \tag{26}$$

The relation $ds/dt > 0$ is readily verified. In the general case, any function $f(q, p, t)$ that is a composite function $f(q, p, t) = f(\rho(q, p, t))$ and satisfies the condition $\Omega(\partial g(\rho))/(\partial \rho) < 0 \, (\forall t)$ is a Lyapunov function, i.e., $(df)/(dt) > 0$. It is important to note that the condition $\Omega > 0$ or $\Omega(\partial g(\rho))/(\partial \rho) < 0$ is not obligatory [57].

4. QUANTUM DISSIPATIVE MECHANICS

We use the usual rule for defining quantum physical variables that have classical analog [18]: If A, B, C are the operators of physical variables a, b, c that satisfy the classical Poisson bracket $[a, b] = c$, then the operators satisfy the commutation relation $[A, B] \equiv (AB) - (BA) = ihC$. Taking into account the characteristic properties of physical variables, we readily obtain relations for the operators of these variables:

$$[Q^i, Q^j] = [P_i, P_j] = 0, \quad [Q^i, P_j] = i\hbar\delta_j^i, \tag{27}$$

$$[W, P_i] = i\hbar W_i^q \quad , \quad [W, Q^i] = -i\hbar W_p^i, \tag{28}$$

$$[[W, P_i], P_j] \neq [[W, P_j], P_i] \quad i \neq j \quad \text{or} \quad J[P_i, W, P_j] = \Omega_{ij} \neq 0, \tag{29}$$

$$[[W, Q^i], Q^j] \neq [[W, Q^j], Q^i] \quad i \neq j, \quad \text{or} \quad J[Q^i, W, Q^j] = \Omega^{ij} \neq 0, \tag{30}$$

$$[Q^i, [W, P_j]] \neq [P_j, [W, Q^i]] \quad \text{or} \quad J[Q^i, W, P_j] = \Omega_j^i \neq 0, \tag{31}$$

where

$$J[A,B,C] = -1/(\hbar^2)\,(\,[A[BC]] + [B[CA]] + [C[AB]]\,)\ \text{and}\ Q^\dagger = Q;\ P^\dagger = P;\ W^\dagger = W;\ \Omega^\dagger = \Omega.$$

We require that the canonical quantum commutation relations be a part of these relations. If we are to have fulfillment of the commutation relations (27)—(31) and the canonical quantum commutation rules, the operator of a nonholonomic variable must be a nonassociative operator. At the same time, it is sufficient to require that the operator W satisfy the following conditions:

1) left and right associativity: $(Z^k, Z^l, W) = (W, Z^k, Z^l) = 0$;

2) left—right nonasssociativity: $(Z^k, W, Z^l) \neq 0$ if $k \neq l$, where $(A, B, C) \equiv (A(BC)) - ((AB)C)$ is the associator, $k, l = 1,\ldots,2n$, $Z^i = Q^i$, and $Z^{n+i} = P_i$, $i = 1,\ldots,n$.

The states in quantum dissipative mechanics can be represented by the operator of a density matrix (density operator) $\rho(t)$ that satisfies the condition $\rho^\dagger(t) = \rho(t)$. The total derivative of the operator of a physical variable $A(t) \equiv A(Q, P, t)$ and of the state operator $\rho(t)$ can be written in the form

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}[H - W, A], \tag{32}$$

$$\frac{d\rho}{dt} = -\frac{1}{2}[\,\rho\,,\,\Omega\,]_+\,, \tag{33}$$

where the anticommutator $[,]_+$ has arisen on account of the Hermiticity of the density operator ρ and the operator Ω. The solution of the first equation can be expressed in the form

$$A(t) = S(t,t_0)A(t_0)S^\dagger(t,t_0),\ \text{where}\ \ S(t,t_0) = T\exp\frac{i}{\hbar}\int_{t_0}^{t} d\tau\,(H - W)(\tau). \tag{34}$$

Here, the T exponential is defined in the usual manner [80], but the following rule is added for the expansion of the exponential of a nonassociative operator:

$$\exp A = 1 + A + \tfrac{1}{2}(AA) + \tfrac{1}{6}((AA)A) + \tfrac{1}{24}(((AA)A)A) + \ldots.$$

The solution of Eq. (33) can be represented in the form

$$\rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0),\ \text{where}\ U(t,t_0) = T\exp\frac{1}{2}\int_{t_o}^{t} d\tau\,\Omega(\tau). \tag{35}$$

Thus, the evolution in time of the operator of a physical variable $A(t)$ is unitary, but the evolution of the state operator $\rho(t)$ is nonunitary. Therefore, a pure state at the time $t = t_0$ $[\rho^2(t_0) = \rho(t_0)]$ becomes a mixed state at a subsequent time $t \neq t_0$, and an entropy operator η for the state $\rho(t)$ [81, 12] can be defined in the form $\eta(t) = -\ln\rho(t)$. The entropy operator satisfies the equation $d\eta(t)/dt = \Omega$.

It can be seen that the commutator of a product of operators and a nonassociative operator W and the total time derivatives of a commutator and a product of two operators do not satisfy Leibniz's rule:

$$[AB, W] = A[B, W] + [A, W]B\ +\ (A, W, B), \tag{36}$$

$$\frac{d}{dt}[A, B] = \left[\frac{d}{dt}A, B\right] + \left[A, \frac{d}{dt}B\right]\ +\ J[A, W, B], \tag{37}$$

$$\frac{d}{dt}(AB) = \left(\left(\frac{d}{dt}A\right)B\right) + \left(A\left(\frac{d}{dt}B\right)\right)\ +\ (A, W, B), \tag{38}$$

where A and B are associative operators (operators of holonomic functions). We define a canonical (unitary) transformation [18] of the operator $A_H(t) = A(t)$ in the form $A_S(t,\ t_0) = S^\dagger(t,\ t_0)A_H(t)S(t,\ t_0)$. The operator $A_S(t,\ t_0)$ satisfies the condition $A_S(t_0,\ t_0) = A_H(t_0)$. In this case, Eqs. (33) and (34) take the form

$$\frac{d}{dt}\rho_S(t,t_0) = \frac{i}{\hbar}[\rho_S,\ (H - W)_S] - \frac{1}{2}[\Omega_S,\ \rho_S]_+\,. \tag{39}$$

This equation is the dissipative analog of the Schrödinger equation, and the operators $A_H(t)$ and $A_S(t)$ are the Heisenberg and Schrödinger representations of the operator $A(t)$. The solution of Eq. (35) can be represented in the form

$$\rho_S(t,t_0) = U_S^{\dagger}(t,t')\rho_S(t',t_0)U_S(t,t') \quad , \tag{40}$$

where

$$U_S^{\dagger}(t,t') = T\exp\frac{-i}{\hbar}\int_{t'}^{t} d\tau \left(H - W - \frac{i\hbar}{2}\Omega\right)_S (\tau,t_0). \tag{41}$$

We consider some important properties of basis vectors [71]. In dissipative quantum mechanics, it must be borne in mind that even in the Heisenberg representation the state operators

$$\rho_H(t) = \sum_a \rho_a[\psi_a,t\rangle_H\langle\psi_a,t]_H$$

and the wave vectors $[\psi, t\rangle_H$ evolve in time, i.e., in contrast to ordinary quantum mechanics $[q, t_1\rangle_H \not\equiv [q, t_2\rangle_H$. Therefore, the basis vectors $\{[q, t\rangle\}$ are defined [18] at fixed time points $t=t_f$:

$$1)\ Q_H(t)\,[q,t\rangle_H = [q,t\rangle_H\, q_f, \quad 2)\ \langle q,t]_H\,[q',t\rangle_H = \delta(q-q'),$$

$$3)\ \int dq\ [q,t\rangle_H\,\langle q,t]_H = 1, \quad 4)\ Q_H(t) = \int dq\ [q,t\rangle_H\, q_f\,\langle q,t]_H$$

$$5)\ [\psi,t\rangle_H = \int dq\ [q,t_f\rangle_H\,\Psi_H(q,t,t_f),$$

where

$$\Psi_H(q,t,t_f) = \langle q,t_f]_H\,[\psi,t\rangle_H\ .$$

It is easy to prove the following propositions:

1) a unitary transformation maps a basis vector to a basis vector;

2) for any two vectors defined at two different time points there exists a unitary transformation that relates them. As a result, the Schrödinger representation of the basis vector $[q, t, t_0\rangle_S \equiv S^{\dagger}(t-t_0)[q, t\rangle_H$ can be regarded as a unitary transformation of the basis vector

$$[q,t_0\rangle_H = S^{\dagger}(t-t_0)\,[q,t\rangle_H = [q,t,t_0\rangle_S.$$

For the indicated reasons, the trace of an operator can be defined only for a fixed time instant. Note that the density operator $\rho(t)$ satisfies the usual condition $\mathrm{Sp}_t(\rho(t))=1$ $(\forall t=t_{\mathrm{fixed}})$ where we have taken into account the fixed-time definition of the basis vectors. The average for a physical variable $A(t)=A(p, q, t)$ is defined in the form

$$A_{av}(t) = \langle A(t)\rangle_t = \mathrm{Sp}_t(A(t)\rho(t))\ (\forall t = t_{\mathrm{fixed}}),$$

and the time derivative of the average of a physical variable can be defined only as the average rate of change of the operator of the given physical variable:

$$\tfrac{d}{dt}\langle A(t)\rangle_\tau \equiv \tfrac{d}{dt}\mathrm{Sp}_\tau(\rho(\tau)A(t))\ \ (\forall t = t_{\mathrm{fixed}}).$$

We consider the Green's function of the Schrödinger equation and its Feynman representation [82]. Taking into account Eq. (33), we can write the dissipative analog of the Schrödinger equation for the wave vector in the form

$$i\hbar\frac{d}{dt}[\psi,t,t_0\rangle_S = \left(H - W - \frac{i\hbar}{2}\Omega\right)_S (t,t_0)\,[\psi,t,t_0\rangle_S\,. \tag{42}$$

The simplest example of this equation for a one-dimensional harmonic oscillator with friction is considered in the Appendix. Taking into account a time dependence of the state vectors in the Heisenberg representation, we shall distinguish the following Green's functions:

$$\Psi_S(q,t) = \int dq'\, G(q,q',t-t')\,\Psi_S(q',t'), \tag{43}$$

$$\Psi_H(q,t) = \int dq'\, G_H(q,q',t-t')\,\Psi_H(q',t'), \tag{44}$$

where

$$\Psi_S(q,t) \equiv \langle q,t]_S[\psi,t\rangle_S \equiv \langle q,t]_H[\psi,t\rangle_H \ ; \tag{45}$$

$$\Psi_H(q,t) \equiv \langle q,t]_H[\psi,t\rangle_S \equiv \langle q,t]_S[\psi,t\rangle_H \ ; \tag{46}$$

$$G_S(q,q',t-t') \equiv \left\langle q,t]_S\, U_S^{\dagger}(t,t')\, q',t' \right\rangle_S \theta(t-t') \equiv \left\langle q,t]_H\, U_H^{\dagger}(t,t')\, [q',t' \right\rangle_H \theta(t-t') \ ; \tag{47}$$

$$G_H(q.q',t-t') \equiv \left\langle q,t]_S\, U_H^{\dagger}(t-t')\, [q',t' \right\rangle_S \theta(t-t') \tag{48}$$

and $[q,\ t\rangle_H \equiv [q,\ t=t_{\text{fixed}}\rangle_H$. It is readily seen that the Green's function satisfies the equation

$$i\frac{d}{dt}G_S(q,q',t) = \left(H - W - \frac{i\hbar}{2}\Omega\right)_S G_S(q,q',t) \text{ and } G_S(q,q',0) = \delta(q-q'). \tag{49}$$

We use Faddeev's method [82] and the conditions

$$\left\langle p,t_f\right]_H \left(H - W - \frac{i\hbar}{2}\Omega\right)_H \left[q,t_f\right\rangle_H = \left(h - w - \frac{i\hbar}{2}\Omega_{cl}\right)(q_f,p_f)\,\langle p,t_f]_H[q,t_f\rangle_H \,, \tag{50}$$

$$\left\langle q^{n+1},t_n]_S U_S^{\dagger}(t_{n+1}-t_n)[q^n,t_n\right\rangle_S$$

$$\simeq \left\langle q^{n+1},t_n\right]_H \exp\frac{-i(t_{n+1}-t_n)}{\hbar}\left(H - W - \frac{i\hbar}{2}\Omega\right)_H (t_n)\left[q^n,t_n\right\rangle_H . \tag{51}$$

The Feynman representation for the Green's function takes the form

$$G_S(q,q',t-t') = \int DqDp \exp\frac{i}{\hbar}\int_{t'}^{t} d\tau\left(p\frac{dq}{d\tau} - h(q,p,\tau) + w(q,p,\tau) + \frac{i\hbar}{2}\Omega_{cl}\right). \tag{52}$$

Similarly, we can define a functional integral and generating functional in quantum field theory [80, 83, 85, 86], which were considered in [71, 77]. As method of solving quantum dissipative problems it has been proposed to use normal geodesic coordinates and the covariant background field method [88—94], which generalize the method of expansion in Taylor series around the classical solutions and are considered for Riemannian [88, 89, 73], affine [90—94], and affine—metric [93, 91, 94, 78, 79] manifolds. This method makes it possible to avoid some difficulties associated with the nonassociative and non-Lie properties of the operator of a nonholonomic variable (the Sedovian) whose variation is linear in the variations of the coordinates [71, 77]. The background field method is a method for obtaining approximations by conservative systems for a quantum dissipative system.

As an example, we consider a multidimensional harmonic oscillator with friction. It is well known that the metric of the configuration space of the harmonic oscillator is determined by the kinetic energy [1]:

$$d^2s = 2T(dt)^2 = \delta_{ij}\,dq^i dq^j, \tag{53}$$

where $T=\Sigma_i^n(1/2)(dq^i/dt)^2$ is the kinetic energy of the harmonic oscillator. The existence of potential forces with potential $U(q)=\Sigma_{i=1}^n(\omega_i^2/2)(q^i)^2$, $\omega_i^2=k_i/m_i$, leads to a deformation of the metric of the configuration space in the form $d^2s=(E-U(q))\,\delta_{ij}dq^i dq^j$ only for conservative systems [1], where E is the total energy. Dissipative forces make it impossible to obtain the mechanical orbit from Jacobi's variational principle [1]. Thus, the configuration space of the harmonic oscillator with friction is flat. The equation of motion for such an oscillator in the n-dimensional configuration space has the form

$$du^i/dt + \omega_i^2 q^i = w_i(q,u), \tag{54}$$

where $i,\ j=1,\ldots,n$; $u^i \equiv dq^i/dt$, and $w_i(q,\ u)$ is the dissipative force. We consider a friction force w_i of the form

$$w_i(q,u) = c_{ij}(q)u^j + D_{ikl}(q)u^k u^l. \tag{55}$$

It is easy to obtain the background-field expansion of the Hamiltonian $h(q,\ p)$, the nonholonomic function w, and $\Omega(q,\ p)$ around the classical solution q_0^i of Eqs. (54) in the form of a series in $\xi^i(t)$: $q^i(t)=q_0^i(t)+\xi^i(t)$. The functional integral (52) with respect to the momenta p^i is a Gaussian integral, and the Feynman path integral for the Green's function is obtained in the form

$$G_S(q,q',t-t') = N\int D\xi^i \exp\frac{i}{\hbar}\int_{t'}^{t} d\tau\,(T(q_0)+Z_1(q_0,\xi,\tau)+Z_2(q_0,\xi,\tau)), \tag{56}$$

where $T(q_0)=\frac{1}{2}\delta_{ij}v_0^iv_0^j$, $v_0^i=dq_0^i/dr$,

$$Z_1(q_0,\xi)=\frac{1}{2}\delta_{ij}\frac{d\xi^i}{d\tau}\frac{d\xi^j}{d\tau} - 2D_{ikj}(q_0)v_0^k\xi^i\frac{d\xi^j}{d\tau} - D_{inm;j}(q_0)v_0^nv_0^m\xi^i\xi^j - \frac{\omega_i^2}{2}(\xi^i)^2 \tag{57}$$

and $a_{n;m}(q)\equiv\partial a_n(q)/\partial q_m$. Here, $Z_2(q_0,\xi)$ is the sum of the terms of the series with respect to the fields ξ that do not contribute to the single-loop redefinition of the metric of the configuration space. In the calculation of the vacuum diagrams, $q^i(t)$ are regarded as external fields of one-dimensional field theory, and $\xi^i(t)$ as quantum fields. The vacuum contribution to the Green's function, which has the form $(1/2)T_{ij}(q_0)v_0^iv_0^j$, leads to a redefinition of the metric:

$$d^2s = 2T(q)(dt)^2 = (\delta_{ij}+T_{ij}(q))dq^idq^j.$$

The single-loop vacuum contribution for the harmonic oscillator with friction (55) has the form

$$T_{ij}=\sum_{k=1}^{n}\left(\sum_{l=1}^{n}\frac{2}{\omega_k+\omega_l}D_{kli}(D_{klj}-D_{lkj})-\frac{1}{\omega_k}D_{ikk;ij}\right). \tag{58}$$

It is readily seen that the background-field configuration space for the quantum harmonic oscillator with friction (55) is not flat. If the friction is quadratic in the velocities with coefficients that depend on the coordinates, or these coefficients are not completely symmetric tensors, then the configuration space is curved by virtue of quantum fluctuations. The complete expression for the two-loop redefinition of the metric is cumbersome, and therefore we give the condition under which the configuration space remains flat with allowance for the single- and two-loop contributions of the vacuum diagrams:

$$D_{ikl;j}=0 \quad \text{and} \quad D_{ikl}=D_{(ikl)}. \tag{59}$$

Finally I should like to thank A. P. Demichev and F. A. Lunev for helpful discussions and all my colleagues at the Department of Theoretical High Energy Physics at the Skobel'tsin Institute of Nuclear Physics at the Moscow State University for support during this work.

APPENDIX

We introduce the Hamiltonian and Sedovian in the form

$$h=\frac{p^2}{2m}+\frac{m\omega^2q^2}{2}\,;\quad \delta w=\gamma mp\,\delta q.$$

They determine a one-dimensional harmonic oscillator with friction. We use the background-field method and expand the Hamiltonian and Sedovian in Taylor series in $Q=q-q_0$, where q_0 is a solution of the classical equation of motion in the configuration space. As solution, we take $q_0=0$ and specify QP ordering of the operators that occur in the expansion of the Sedovian. In the given case, the dissipative analog of the Schrödinger equation takes the form

$$i\hbar\frac{d}{dt}\Psi(t)=\left[-\frac{\hbar}{2m}\frac{\partial^2}{\partial Q^2}+i\hbar\gamma\,Q\frac{\partial}{\partial Q}+\frac{m\omega^2}{2}Q^2-\frac{i}{2}\gamma\right]\Psi(t).$$

The stationary state

$$\Psi(\xi,t)=u(\xi)\exp-\tfrac{i}{\hbar}Et$$

is determined by the equation

$$u''(\xi)-a\xi u'(\xi)+(\varepsilon-\xi^2)u(\xi)=0,$$

where

$$a=\frac{2i\gamma}{\omega}\,;\ \xi=\sqrt{\frac{m\omega}{\hbar}}Q\,;\ \varepsilon=\frac{2}{\hbar\omega}(E-\frac{i}{2}\gamma).$$

We consider a function $u(\xi)$ in the form

$$u(\xi) = \left(\sum_{k=0}^{n} A_k \xi^k \right) \exp -\frac{1}{2} s\xi^2,$$

where s is a solution of the equation $s^2+as-1=0$ and $n<\infty$. As a result, we obtain the eigenvalues

$$E_n = \hbar\sqrt{\omega^2-\gamma^2}(n+\tfrac{1}{2}) - i\gamma$$

for $0<\gamma^2/\omega^2<\frac{1}{2}$ and a continuous spectrum for $\gamma^2/\omega^2>\frac{1}{2}$. Note that the lifetime of a state is $T=\hbar/2\gamma<\infty$. The results can be rewritten in the form

$$\Delta E_n(\omega) = (\hbar\sqrt{\omega^2-\gamma^2} \text{ for } \omega^2 > 2\gamma^2) \bigwedge (0 \text{ for } \omega^2 < 2\gamma^2).$$

We note that the discontinuity at the point $\omega=\sqrt{2}\gamma$ is a purely quantum dissipative effect. The obtained eigenvalues demonstrate the correctness of Kanai's approach [32], but restrict the domain of applicability to the background-field approximation and the corresponding ordering of the operators Q and P.

REFERENCES

1. C. Lanczos, *The Variational Principles of Mechanics*, Toronto University Press (1949).
2. H. Goldstein, *Classical Mechanics*, Addison-Wesley (1950).
3. L. I. Sedov, "Applied mechanics," in: *Proc. of the 11th Intern. Congr. Appl. Mech., Munich, 1964*, Springer-Verlag (1966), p. 9.
4. L. I. Sedov, *Usp. Mat. Nauk*, **20**, 121 (1965).
5. L. I. Sedov, "Irreversible aspects of continuum mechanics and transfer of physical characteristics of moving fluids," in: *Proc. IUTAM Symp., Vienna, 1966*, Springer-Verlag (1968), p. 346.
6. L. I. Sedov, *Z. Angew. Math. Phys.*, **20**, 643 (1969).
7. L. I. Sedov, *Prikl. Mat. Mekh.*, **32**, 771 (1968).
8. L. I. Sedov and A. G. Tsypkin, *Principles of the Macroscopic Theory of Gravitation and Electromagnetism* [in Russian], Nauka, Moscow (1989).
9. H. Poincaré, *Acta Math.*, **13**, 67 (1890); *Rev. Metaphys. Morale*, **1**, 534 (1893).
10. E. Zermelo, *Ann. Phys. (N.Y.)*, **57**, 485 (1896).
11. B. Misra, *Proc. Nat. Acad. Sci. USA.*, **75**, 1629 (1978).
12. I. Prigogine, *From Being to Becoming*, W. H. Freeman & Co., San Franscico (1980).
13. N. N. Bogolyubov, "Problems of a dynamical theory in statistical physics," in: *Studies in Statistical Mechanics*, Vol. 1 (eds. J. de Boer and G. E. Uhlenbeck), North-Holland, Amsterdam (1962).
14. N. N. Bogolyubov, *Zh. Eksp. Teor. Fiz.*, **16**, 691 (1946).
15. R. L. Liboff, *Introduction to the Theory of Kinetic Equations*, New York (1969).
16. R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics*, Wiley-Interscience, New York (1975).
17. P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series No. 2, Yeshiva University, New York (1964).
18. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford (1958).
19. E. P. Wigner, *Symmetries and Reflections, Scientific Essays*, MIT Press (1970).
20. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton (1955).
21. R. Penrose, *The General Theory of Relativity* [Russian translation], Mir, Moscow (1983), p. 233.
22. S. Hawking, *Black Holes* [Russian translation], Mir, Moscow (1978), p. 169.
23. S. W. Hawking, *Commun. Math. Phys.*, **87**, 395 (1982).
24. S. Hawking, *The General Theory of Relativity* [Russian translation], Mir, Moscow (1983), p. 363.
25. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
26. J. Ellis, J. S. Hagelin, D. V. Nanopoulos, and M. Srednicki, *Nucl. Phys.*, **B241**, 381 (1984).
27. R. E. A. C. Paley and N. Weiner, *Fourier Transforms in the Complex Domain*, AMS, New York (1934).
28. M. Namiki and N. Mugibayashi, *Prog. Theor. Phys.*, **10**, 474 (1953).
29. R. W. Hasse, *J. Math. Phys.*, **16**, 2005 (1975).
30. C. G. Callan and Thorlacius, *Nucl. Phys.*, **B329**, 117 (1990).
31. H. Bateman, *Phys. Rev.*, **38**, 815 (1931).
32. E. Kanai, *Prog. Theor.Phys.*, **3**, 440 (1948).
33. W. H. Brittin, *Phys. Rev.*, **77**, 396 (1950).
34. N. A. Lemos, *Phys. Rev.*, D **24**, 2338 (1981).
35. H. V. Helmholtz, *Borchardt-Crelle J. Mathem.*, **100**, 137, 213 (1886); *Wiss. Abh. von H. V. Helmholtz*, Vol. 3, Leipzig (1895), p. 203.
36. *The Variational Principles of Mechanics* [in Russian], Nauka, Moscow (1959).

37. P. Havas, *Bull. Am. Phys. Soc.*, **1**, 337 (1956).
38. P. Havas, *Nuovo Cimento. Ser. 10. Suppl.*, **5**, 363 (1957).
39. I. K. Edwards, *Am. J. Phys.*, **47**, 153 (1979).
40. S. A. Hojman and L. C. Shepley, *J. Math. Phys.*, **31**, 142 (1991).
41. G. Lindblad, *Commun. Math. Phys.*, **48**, 119 (1976).
42. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.*, **17**, 821 (1976).
43. V. Gorini, A. Frigerio, M. Verri, Kossakowski, and E. C. G. Sudarshan, *Rep. Math. Phys.*, **13**, 149 (1978).
44. O. Bratteli, *Lecture Notes in Mathematics*, Vol. 1055, Springer-Verlag (1984), p. 46.
45. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vols. 1 and 2, New York (1979, 1981).
46. J. Messer and B. Baumgartner, *Z. Phys.*, B **32**, 103 (1978).
47. G. P. Beretta, E. P. Gyftopoulos, J. L. Park, and G. N. Hatsopoulos, *Nuovo Cimento*, **B82**, 169 (1984).
48. G. P. Beretta, E. P. Gyftpopoulos, and J. L. Park, *Nuovo Cimento*, **B87**, 77 (1985).
49. H. J. Korsch and H. Steffen, *J. Phys.*, A **20**, 3787 (1987).
50. M. Hensel and H. J. Korsch, *J. Phys.*, A **25**, 2043 (1992).
51. N. Gisin, *J. Phys.*, A **14**, 2259 (1981); A **19**, 205 (1983); *J. Math. Phys.*, **24**, 17779 (1983).
52. M. P. Kostin, *J. Chem. Phys.*, **57**, 3589 (1972); *J. Stat. Phys.*, **12**, 145 (1975).
53. D. Schuch, K. M. Chung, and H. Hartman, **24**, 1652 (1983).
54. J. Liouville, *J. Math. Pure. Appl.*, **3**, 342 (1838).
55. V. I. Arnol'd, *Ordinary Differential Equations*, 3rd ed. [in Russian], Nauka, Moscow (1984).
56. J. Fronteau, *Hadronic J.*, **2**, 724 (1979).
57. V. H. Steeb, *Physica*, **A95**, 181 (1979).
58. R. M. Santilli, *Hadronic J.*, **1**, 223, 574, 1279 (1978); **2**, 1460 (1979); **3**, 854 (1980).
59. R. Mignani, H. C. Myung, and R. M. Santilli, *Hadronic J.*, **6**, 1873 (1983).
60. R. M. Santilli, *Lie-Admissible Approach to the Hadronic Structure*, Vol. 2, Nonautum: Hadronic Press (1982).
61. R.M. Santilli, *Foundations of Theoretical Mechanics*, Vol. 2, Springer-Verlag (1983).
62. S. Okubo, in: *Proc. of the Third Workshop on Current Problems in High Energy Particle Theory. Florence, Italy*, Johns Hopkins University Press., Baltimore (1979), p. 103.
63. R. Mignani, *Hadronic J.*, **12**, 167 (1989).
64. L. I. Sedov and A. G. Tsypkin, *Prikl. Mat. Mekh.*, **43**, 387 (1979).
65. V. L. Berdichevskii, *Prikl. Mat. Mekh.*, **30**, No. 3, 510 (1966); **30**, No. 6, 1081 (1966); **43**, No. 4, 664 (1979).
66. V. A. Zhelnorovich, *Dokl. Akad. Nauk SSSR*, **176**, 283 (1967); **184**, 55 (1969).
67. L. I. Sedov and V. A. Zhelnorovich, *Prikl. Mat. Mekh.*, **42**, 771 (1978).
68. L. T. Chernyi, *Models of Continuous Media* [in Russian], Nauka, Moscow (1983).
69. M. Planck, *Acht Vorlesungen über theoretische Physik, gehalten an der Columbia University in the City of New York in Frühjahr 1909*, Leipzig (1910).
70. W. Pauli, *Theory of Relativity*, Pergamon Press, Oxford (1958).
71. V. E. Tarasov, "Dissipative quantum dynamics and nonlinear sigma-model," Preprint of Nucl. Phys., 92-33/282, Institute of Moscow State University, Moscow (1992), p. 1.
72. D. Friedan, *Phys. Rev. Lett.*, **45**, 1057 (1980); *Ann. Phys. (N.Y.)*, **163**, 318 (1985).
73. L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *Ann. Phys. (N.Y.)*, **134**, 85 (1981).
74. S. P. De Alwis, *Phys. Rev.*, D **34**, 3760 (1986).
75. G. M. Shore, *Nucl. Phys.*, **B286**, 346 (1987).
76. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge (1987).
77. V. E. Tarasov, in: *Proc. of the Seventh Intern. Workshop on High Energy Physics and Quantum Field Theory*, 7—14 October, Sochi (1992); *Yad. Fiz.*, **56**, 269 (1993).
78. V. V. Belokurov and V. E. Tarasov, *Teor. Mat. Fiz.*, **78**, 471 (1989).
79. V. V. Belokurov and V. E. Tarasov, Preprint IC-90-168, ICTP, Trieste (1990), p. 1.
80. N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, 3rd ed., Wiley, New York (1980).
81. D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, Consultants Bureau, New York (1974).
82. L. D. Faddeev and A. A. Slavnov, *Gauge Fields, Introduction to Quantum Theory*, transl. of 1st Russ. ed., Benjamin/Cummings, Reading, Mass. (1980).
83. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982).
84. G. A. Vilkovisky, *Nucl. Phys.*, **B234**, 125 (1984).
85. B. S. DeWitt, *Dynamical Theory of Groups and Fields*, Gordon and Breach, New York (1965).
86. P. S. Howe, G. Papadopoulos, and K. S. Stelle, *Nucl. Phys.*, **B296**, 26 (1988).
87. J. Honerkamp, *Nucl. Phys.*, **B36**, 130 (1972).
88. G. D. Birkhoff, *Relativity and Modern Physics*, Horvard University Press, Cambridge (1923).
89. L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton (1965).

90. O. Veblen, *Proc. Natl. Acad. Sci. USA*, **8**, 192 (1922).
91. O. Veblen and T. Y. Thomas, *Trans. Am. Math. Soc.*, **25**, 551 (1923); **26**, 373 (1923).
92. V. Hlavaty, *Ann. Soc. Pol.*, **5**, 44 (1926); *Mem. Sci. Mathem. Paris*, **63**, 1 (1934).
93. O. Veblen, *Invariants of Quadratic Differential Forms*, London (1927).
94. T. Y. Thomas, *Differential Invariants of the Generalized Spaces*, Cambridge (1934).