

QUANTUM DISSIPATIVE SYSTEMS. III. DEFINITION AND ALGEBRAIC STRUCTURE

V. E. Tarasov¹

Starting from the requirement of a consistent quantum description of dissipative (non-Hamiltonian) systems, which is formulated as the absence of a contradiction between the evolution equations for quantum dissipative systems and quantum commutation relations, we show that the Jacobi identity is not satisfied. Thus, the requirement for a consistent quantum description forces one to go beyond the Lie algebra. As a result, anticommutative non-Lie algebras are necessary to describe dissipative (non-Hamiltonian) systems in quantum theory.

1. Introduction

In Part I of the present paper [1], we suggested that an anticommutative non-Lie algebra with the Lie algebra as its subalgebra be used in the quantum description of dissipative systems. Note that the necessity of going beyond the Lie algebras in describing quantum dissipative systems is due to problems in constructing their self-consistent descriptions. One of the problems stems from an inconsistency between the quantum commutation relations and the motion equations for a quantum dissipative system. A simple example of this inconsistency is as follows.

The quantum commutation relations

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

are inconsistent with the Langevin equations [2]

$$\dot{a} = (-i\alpha - \beta)a + f(t), \quad \dot{a}^\dagger = (i\alpha - \beta)a^\dagger + f(t)^\dagger,$$

where the dot indicates the time derivative and β is the coefficient describing the dissipation (damping) in the system. Taking the commutators from the Langevin equations, we obtain

$$[\dot{a}, a^\dagger] + [a, \dot{a}^\dagger] = -2\beta.$$

On the other hand, differentiating the first of the quantum commutation relations with respect to time and using the Leibnitz rule, we arrive at the identity

$$[\dot{a}, a^\dagger] + [a, \dot{a}^\dagger] = 0.$$

Thus, quantum commutation relations and the evolution equations for the system are consistent only if the system is not dissipative ($\beta = 0$).

One of the methods for solving problems in the quantum description of dissipative systems, proposed in [1, 3], consists in adding a nonassociative operator W to the Heisenberg-Weyl algebra. In [1], we obtained the main properties and commutation relations that the operator W must obey. It turned out that W must be a nonassociative and non-Lie (not obeying the Jacobi identity) operator for all of the commutation

¹Skobeltsyn Institute of Nuclear Physics, Moscow State University, e-mail: TARASOV@THEORY.NPI.MSU.SU.

relations to be valid. As the operator W , which describes the evolution of the quantum dissipative system, is nonassociative and non-Lie, the action of the total time derivative on the product and the commutator of the associative operators do not obey the term-by-term differentiation rule (Leibnitz rule), as the Leibnitz rule is deformed by the appearance of the associator and the algebraic Jacobian operator W , respectively. This removes the contradiction between the quantum equations of motion for dissipative systems and the canonical commutation relations of the Heisenberg-Weyl algebra.

In addition, in [1, 3], we proposed a generalization of the evolution equation for the system state (von Neumann equation) which, unlike others, is a quantum analog of the classical Liouville equation for dissipative systems and, thus, satisfies the correspondence principle.

2. Definition of a dissipative system and the necessity of going beyond Lie algebras

Let us examine, in more detail, the reasons why one must go beyond Lie algebras when constructing the dynamic description of quantum dissipative systems. To this end, we first define a quantum dissipative system.

2.1. Classical dissipative system. The evolution equation for a dynamic system with a finite number of degrees of freedom has the form

$$\frac{dX_k}{dt} = F_k(X). \quad (1)$$

The system is called *dissipative* if at least one of the expressions

$$\Omega_{kl} = \frac{\partial F_k}{\partial X_l} - \frac{\partial F_l}{\partial X_k} \quad (2)$$

differs from zero, and *conservative* if all $\Omega_{kl} = 0$.

Let the evolution of a mechanical system in time, in the phase space of generalized coordinates q_k and momenta p_k , be given by a system of $2n$ ordinary first-order differential equations of the form

$$\frac{dq_k}{dt} = -G_k(t, q, p), \quad \frac{dp_k}{dt} = F_k(t, q, p). \quad (3)$$

A mechanical system is called *Hamiltonian* if the right-hand sides of Eqs. (3) satisfy the following conditions:

$$\Omega^s_{kl}(t, q, p) = 0, \quad (4)$$

where $s = 1, 2, 3$, $k, l = 1, \dots, n$, and

$$\Omega^1_{kl} = \frac{\partial G_l}{\partial p_k} - \frac{\partial G_k}{\partial p_l}, \quad \Omega^2_{kl} = \frac{\partial G_k}{\partial q_l} - \frac{\partial F_l}{\partial p_k}, \quad \Omega^3_{kl} = \frac{\partial F_l}{\partial q_k} - \frac{\partial F_k}{\partial q_l}. \quad (5)$$

In this case, the motion equations for system (3) can be represented as canonical Hamilton equations: hence, they are fully characterized by the system Hamiltonian h . Equations (4) and (5) are analogs of the Helmholtz conditions (4) in the phase space for differential equations (3).

Note that Eqs. (3) can be derived from the stationary principle for the action if and only if conditions (4), (5) are fulfilled. In this case, Eqs. (3) admit various formulations in the class of holonomic functionals called Hamiltonian actions, which are sets of its critical points coinciding with the set of solutions of canonical Hamilton equations. Thus, Eqs. (3) admit the variational principle of the Hamiltonian action stationarity under quite severe limitations (4) on the right-hand side structure.

If evolution equations (3) are such that their right-hand sides do not meet at least one of conditions (4) (and, thus, the equations cannot be derived from the stationary principle of the holonomic functional), the dynamic system is called *dissipative* or *non-Hamiltonian*.

To extend the definition of a dissipative system to quantum theory, one must write the motion equations and the conditions for the non-Hamiltonian (dissipative) property using the Poisson brackets below:

$$\{A, B\} = \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k}. \quad (6)$$

Let us consider a dynamic system, where the evolution equations have the form

$$\frac{dq_k}{dt} = \{q_k, h\} - G_k(t, q, p), \quad \frac{dp_k}{dt} = \{p_k, h\} + F_k(t, q, p). \quad (7)$$

Using Poisson bracket (6), one can write (5) as

$$\Omega^1_{kl} = \{q_k, G_l\} - \{q_l, G_k\}, \quad \Omega^2_{kl} = \{G_k, p_l\} - \{q_k, F_l\}, \quad (8)$$

$$\Omega^3_{kl} = \{p_l, F_k\} - \{p_k, F_l\}. \quad (9)$$

Thus, we can formulate the following definition.

Definition 2.1. The dynamic system whose evolution is given by Eqs. (7) or (3) is called *dissipative* (*non-Hamiltonian*) if at least one of the expressions (8), (9) differs from zero.

It is convenient to consider dissipative systems for which $G_k = 0$. This constraint not only appreciably simplifies many relations, but it is also necessary, from physical considerations, for "relating the Hamiltonian $h(t, q, p)$ to the system energy" [5]. Note that the quantities $F_k(t, q, p)$ physically describe a dissipative system that acts on a mechanical system. In the simplest case, corresponding to the linear dependence of the dissipative forces (resistance forces) on the velocity, they have the form $F_k(t, q, p) = \kappa p_k$, where κ is the coefficient describing the dissipation in the system.

For a dissipative system, the evolution equation for the observable A , which is the function of the generalized coordinates q_k and momenta p_k , has the form

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, h\} + D_0(A), \quad (10)$$

where

$$D_0(A) = \left(F_k \frac{\partial A}{\partial p_k} - \frac{\partial A}{\partial q_k} G_k \right).$$

Note that in the case of a dissipative system, the operator $D_0(A)$ cannot be represented as $\{A, h'\}$, where h' is a function of the generalized coordinates and momenta.

2.2. Quantum dissipative system. Evolution of a quantum system in time is defined by the relation

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar} [H, A], \quad (11)$$

where H is a self-adjoint associative operator called a Hamiltonian and $[A, B] = AB - BA$ is the commutator of the operators A and B . Quantum systems, whose motion equations have form (11), are normally called *Hamiltonian* systems.

Let the evolution of the observable A , which is a function of the operators of coordinates Q_k and momenta P_k , be described for a dissipative system by the equation

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar} [H, A] + D(A), \quad (12)$$

where the operator $D(A)$ cannot be represented as $\frac{i}{\hbar}[A, W]$ and W is the associative Hermitian operator. The operator $D(A)$ is normally called the dissipation operator of A . If there are quantities for which the dissipation operator is different from zero among the dynamic variables of the system, these systems are called *non-Hamiltonian* or *dissipative systems*.

Consider a quantum system where the evolution equations for the operators of coordinates Q_k and momentums P_k have the form

$$\frac{dQ_k}{dt} = \frac{i}{\hbar}[H, Q_k] - G_k(t, Q, P), \quad \frac{dP_k}{dt} = \frac{i}{\hbar}[H, P_k] + F_k(t, Q, P), \quad (13)$$

where

$$G_k(t, Q, P) = -D(Q_k) \quad \text{and} \quad F_k(t, Q, P) = D(P_k). \quad (14)$$

Generalizing Definition 2.1 of the classical dissipative system, we have the following definition.

Definition 2.2. The quantum system whose evolution is given by Eqs. (13) is called a *quantum dissipative system* if the right-hand sides of (13) do not satisfy at least one of the conditions below:

$$\Omega^s_{kl}(t, Q, P) = 0, \quad (15)$$

where $s = 1, 2, 3$, $k, l = 1, \dots, n$, or

$$\Omega^1_{kl} = -\frac{i}{\hbar}([Q_k, G_l] - [Q_l, G_k]), \quad \Omega^2_{kl} = -\frac{i}{\hbar}([G_k, P_l] - [Q_k, F_l]), \quad (16)$$

$$\Omega^3_{kl} = -\frac{i}{\hbar}([P_l, F_k] - [P_k, F_l]). \quad (17)$$

If evolution equations (13) of the quantum system are such that their right-hand sides satisfy all of conditions (15)–(17), this system is called a *Hamiltonian system*.

2.3. Non-Lie nonassociative operator. A quantum description is often a difficult task because the dissipation operator $D(A)$ cannot be represented as $\frac{i}{\hbar}[A, W]$, where W is the associative operator. In [1], we proposed giving up the requirement that the operator W be associative and, thus, the assumption that the Leibnitz rule and the Jacobi identity are valid for quantum dissipative systems.

Assume that the dissipation operator $D(A)$ for A can be represented as $D(A) = \frac{i}{\hbar}[A, W]$. Then, the operators $G_k(t, Q, P)$ and $F_k(t, Q, P)$, specifying the evolution (13) of the quantum dissipative system and defined in (14), can be written as

$$F_k(t, Q, P) = -\frac{i}{\hbar}[W, P_k], \quad G_k(t, Q, P) = \frac{i}{\hbar}[W, Q_k]. \quad (18)$$

In this case, (16) and (17) can be represented as

$$\Omega^1_{kl} = J[Q_k, Q_l, W] = \frac{1}{\hbar^2}([[W, Q_k], Q_l] - [[W, Q_l], Q_k]) \quad (k \neq l), \quad (19)$$

$$\Omega^2_{kl} = J[Q_k, P_l, W] = \frac{1}{\hbar^2}([[W, Q_k], P_l] - [[W, P_l], Q_k]), \quad (20)$$

$$\Omega^3_{kl} = J[P_k, P_l, W] = \frac{1}{\hbar^2}([[W, P_k], P_l] - [[W, P_l], P_k]) \quad (k \neq l), \quad (21)$$

where

$$J[A, B, C] = \frac{1}{\hbar^2}([[A, B], C] + [[B, C], A] + [[C, A], B])$$

is the Jacobian of the operators A, B, C .

Thus, the definition of the quantum dissipative system as a system where at least one of the conditions (15)–(17) is not fulfilled can be reformulated in terms of Jacobians (19)–(21). The quantum system in which the evolution of observables $A = A(t, Q, P)$ is described by the equation

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}[H - W, A] \quad (22)$$

is called a quantum dissipative (non-Hamiltonian) system if at least one of the expressions for Ω_{kl}^s , describing Jacobians (19)–(21) of the operators Q_k , P_k and W , differs from zero. Thus, we arrive at the following definition.

Definition 2.3. Let the dynamic variables (observables) of a quantum system be described by the associative self-adjoint operators $A = A(t, X)$, whose evolution is given by the equation

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}[\mathcal{H}, A], \quad (23)$$

where \mathcal{H} is an operator that characterizes the given system ($\mathcal{H} = H - W$), and X are the operators of coordinates Q_k and momenta P_k . This dynamic system is called a *quantum dissipative system* if at least one of the Jacobians $J[X_k, X_l, \mathcal{H}]$ of the operators Q_k , P_k and \mathcal{H} differs from zero.

Thus, we have progressed from the requirement that a self-consistent quantum description of dissipative systems be formulated as the absence of contradiction between the evolution equations for quantum dissipative systems and the quantum commutation relations. As a result, the operator that describes the evolution of the quantum dissipative system (the generalized Hamiltonian \mathcal{H}) must violate the Jacobi identity. Consequently, one has to use anticommutative non-Lie algebras to describe quantum dissipative systems.

Note that, generally speaking, the rule of term-by-term differentiation with respect to time (the Leibnitz rule) does not apply to quantum dissipative systems (see the Appendix). This is one of the essential differences between dissipative and Hamiltonian systems.

The use of non-Lie algebras results in deformation of the term-by-term differentiation rule for quantum dissipative systems. For example, the action of the total time derivative on the product of the operators deforms the Leibnitz rule because of the appearance of the following associators:

$$D(AB) = D(A)B + AD(B) + Z(A, B), \quad (24)$$

where

$$Z(A, B) = (A, B, W) - (A, W, B) + (W, A, B),$$

$(x, y, z) \equiv (xy)z - x(yz)$ is the associator of the operators x, y, z , and $D = -i\hbar(d/dt) = [\mathcal{H}, \cdot]$. The actions of the total time derivative on the commutator of the operators also results in deformation of the Leibnitz rule because an algebraic Jacobian of the operators A, B , and W , below, arises:

$$D([A, B]) = [D(A), B] + [A, D(B)] + J(A, B), \quad (25)$$

where

$$J(A, B) = \hbar^2 J[A, \mathcal{H}, B] = \hbar^2 J[A, B, W] = Z(A, B) - Z(B, A).$$

Relations (24) and (25), generalizing the Leibnitz rules, remove the contradiction between the evolution equations for quantum dissipative systems (13) and the quantum commutation relations (see Statement A.1 in the Appendix).

2.4. State evolution equation for a quantum dissipative system. An important property of dissipative processes is the entropy variation. Nevertheless, the quantum equation for the evolution of the density operator $\rho(t, Q, P)$ (the von Neumann equation)

$$\frac{\partial}{\partial t}\rho = -\frac{i}{\hbar}[H, \rho] \quad (26)$$

retains the entropy

$$\langle S \rangle = \text{Sp}(\rho \log \rho)$$

unchanged. Therefore, to describe quantum dissipative systems, one normally uses a generalization of (26), which has the form

$$\frac{\partial}{\partial t}\rho = -\frac{i}{\hbar}[H, \rho] + K(\rho). \quad (27)$$

The operator $K(\rho)$ describes the dissipative part of the density operator evolution in time.

Different authors [6, 7] have examined different forms of the operator $K(\rho)$. However, the generalizations proposed for (26) are not related to the classical Liouville equation for dissipative systems [8, 1],

$$\frac{\partial}{\partial t}\rho = -\{\rho, h\} - D_0(\rho) + \Omega\rho, \quad (28)$$

where

$$D_0(\rho) = \left(F_k \frac{\partial \rho}{\partial p_k} - \frac{\partial \rho}{\partial q_k} G_k \right), \quad (29)$$

$$\Omega = \sum_{k=1}^n \Omega^2_{kk} = \sum_{k=1}^n (\{G_k, p_k\} - \{q_k, F_k\}) = \sum_{k=1}^n \left(\frac{\partial G_k}{\partial q_k} - \frac{\partial F_k}{\partial p_k} \right). \quad (30)$$

The quantum analog of the Liouville equation (28) has the form

$$\frac{\partial}{\partial t}\rho(t, Q, P) = -\frac{i}{\hbar}[H, \rho] + D(\rho) + \frac{1}{2}(\Omega\rho + \rho\Omega), \quad (31)$$

where $\Omega(Q, P) = \sum_{k=1}^n \Omega^2_{kk}$ and Ω^2_{kk} are defined by (16). On the assumption that the dissipation operator can be represented as $D(\rho) = \frac{i}{\hbar}[\rho, W]$, we obtain the quantum equation of the density operator evolution for a dissipative system (generalization of the von Neumann equation) in the form [1, 3]

$$\frac{\partial}{\partial t}\rho(t, Q, P) = -\frac{i}{\hbar}[H - W, \rho] + \frac{1}{2}(\Omega\rho + \rho\Omega), \quad (32)$$

where

$$\Omega(Q, P) = \sum_{k=1}^n \Omega^2_{kk} = \sum_{k=1}^n J[Q_k, P_k, W]. \quad (33)$$

Unlike other proposed equations, this generalization of the von Neumann equation is directly derived from the Liouville equation for dissipative systems (28)–(30) and is its quantum analog.

As a result, we can introduce the definition of a quantum dissipative system based on the evolution equations of the dynamic state system rather than on the evolution equations of the observables.

Definition 2.4. Let the state of the quantum system be described by a self-adjoint density operator $\rho = \rho(t, Q, P)$, whose evolution in time is given by the equation

$$\frac{\partial}{\partial t} \rho(t, Q, P) = -\frac{i}{\hbar} [\mathcal{H}, \rho] + \frac{1}{2} (\Omega \rho + \rho \Omega), \quad (34)$$

where \mathcal{H} is an operator characterizing the given system. Then the dynamic system is called a *quantum dissipative system* if at least one of the Jacobians $J[Q_k, \mathcal{H}, Q_l]$, $J[Q_k, \mathcal{H}, P_l]$, or $J[P_k, \mathcal{H}, P_l]$ differs from zero and the operator Ω can be represented as

$$\Omega(Q, P) = \sum_{k=1}^n J[Q_k, \mathcal{H}, P_k]. \quad (35)$$

Thus, one can proceed from the requirement that self-consistent quantum descriptions of dissipative systems be formulated as the inter-relation of the state evolution equation for the quantum dissipative system and the Liouville equation for the classical dissipative system. However, this still results in a violation of the Jacobi identity by the evolution operator, i.e., anticommutative non-Lie algebras are necessary for describing quantum dissipative systems.

3. Generalized Heisenberg–Weyl algebra

3.1. Heisenberg–Weyl algebra. Below we use a system of units where the Planck constant is equal to unity ($\hbar = 1$) and $G_k(t, Q, P) = i[W, Q_k]$ is assumed to be equal to zero.

Recall that the *Heisenberg–Weyl algebra* W_N is a real $(2N + 1)$ -parametric Lie algebra given by the permutation relations

$$[e^{(1)}, e_k^{(2)}] = [e^{(1)}, e_k^{(3)}] = [e_k^{(2)}, e_l^{(2)}] = [e_k^{(3)}, e_l^{(3)}] = 0, \quad [e_k^{(2)}, e_l^{(3)}] = \delta_{kl} e^{(1)}, \quad (36)$$

where $k, l = 1, 2, \dots, N$. In quantum theory, one often uses the basic elements

$$I = ie^{(1)}, \quad Q_k = ie_k^{(2)}, \quad P_k = ie_k^{(3)}, \quad (37)$$

which are interpreted as the unit, coordinate, and momentum operators, respectively. The basic elements $\{I, Q_k, P_l\}$ obey the permutation relations

$$[I, Q_k] = [I, P_k] = [Q_k, Q_l] = [P_k, P_l] = 0, \quad [Q_k, P_l] = i\delta_{kl} I, \quad (38)$$

which are called canonical commutation relations. Expressions (38) mean that the operators $\{I, Q_i, P_i\}$ generate a Lie algebra called the Heisenberg–Weyl algebra. The generic element of the Heisenberg–Weyl algebra has the form

$$z_0 = x_k^{(1)} e_k^{(1)} + x_k^{(2)} e_k^{(2)} + x_k^{(3)} e_k^{(3)} = sI + x_k Q_k + y_k P_k.$$

3.2. Generalized Heisenberg–Weyl algebra. We introduce the following definition.

Definition 3.1. The *generalized Heisenberg–Weyl algebra* W_N^* is a real $2(N + 1)$ -parametric algebra given by permutation relations (36) and the relations

$$[e^{(1)}, e^{(4)}] = [e_k^{(2)}, e^{(4)}] = 0, \quad [e_k^{(3)}, e^{(4)}] = F_k(e^{(2)}, e^{(3)}), \quad (39)$$

where $F_k(\cdot)$ is a function of the basic elements $e_k^{(2)}$ and $e_k^{(3)}$. In the simplest case of the linear generalized Heisenberg–Weyl algebra LW_N^* , the second relation of (39) has the form

$$[e_k^{(3)}, e^{(4)}] = e_k^{(3)}. \quad (40)$$

With the basic elements $\{I, Q_k, P_k, W\}$, defined by (37), and $W = ie^{(2)}$, permutation relations (36) take the form

$$[I, W] = [Q_k, W] = 0, \quad [W, P_k] = \imath F_k(Q, P). \quad (41)$$

In (41), the element $F_k(Q, P)$ is not basic, but there is a function of the basic elements Q_k, P_k . The element $F_k(Q, P)$ can be considered to mean a polynomial of the basic elements Q_k, P_k . In the simplest case, which is of interest in terms of practical applications, the second relation of (41) has the form

$$[W, P_k] = \imath P_k. \quad (42)$$

The generalized Heisenberg–Weyl algebra W_N^* corresponds to a dissipative system with N degrees of freedom. The physical meaning of the element $F_i(Q, P)$ is that it describes the dissipative force (friction) acting on the system in question. Case (42) corresponds to the linear dependence of the dissipative forces on velocity. Note that the relation $[W, Q_i] = 0$ stems from the physical requirement of the “relation between the Hamiltonian and the system energy” [5, 1].

3.3. Jacobian identities.. The consequences of the generalized commutation relations (38) and (41) are the following identities for the Jacobians:

$$J[Q_k, Q_l, Q_j] = J[Q_k, Q_l, P_j] = J[Q_k, P_l, P_j] = J[P_k, P_l, P_j] = 0, \quad (43)$$

$$J[W, W, W] = J[Q_k, W, W] = J[P_k, W, W] = J[Q_k, Q_l, W] = 0, \quad (44)$$

$$J[P_k, P_l, W] = \imath([F_k, P_l] - [F_l, P_k]), \quad J[Q_k, P_l, W] = -\imath[F_l, Q_k], \quad (45)$$

where $J[x, y, z] = [[x, y], z] + [[y, z], x] + [[z, x], y]$ is the Jacobian of the elements x, y, z .

Some Jacobians of the basic elements (45) are seen to differ from zero by virtue of permutation relations (41), which is the main feature distinguishing dissipative and nondissipative systems. The difference of Jacobians (45) from zero indicates that the generalized Heisenberg–Weyl algebra W_N^* is not a Lie algebra.

3.4. Generic element of the generalized Heisenberg–Weyl algebra. Consider the generic element z of the generalized Heisenberg–Weyl algebra W_N^* in a field of real or complex numbers

$$z = sI + x_k Q_k + y_k P_k + tW, \quad (46)$$

where s, x_k, y_k, t are numbers. The commutator of two elements z_1 and z_2 has the form

$$[z_1, z_2] = \imath s_3 I + \imath t_k F_k(Q, P), \quad (47)$$

where

$$s_3 = x_k^1 y_k^2 - x_k^2 y_k^1, \quad t_k = t^1 y_k^2 - t^2 y_k^1.$$

We represent the Jacobian of three arbitrary elements z_1, z_2 , and z_3 as

$$J[z_1, z_2, z_3] = \imath s_{kl} [Q_k, F_l] + \imath t_{:l} ([F_k, P_l] - [F_l, P_k]), \quad (48)$$

where

$$s_{kl} = (x_k^1 y_l^2 - x_k^2 y_l^1) t^3 + (x_k^1 y_l^3 - x_k^3 y_l^1) t^2 + (x_k^2 y_l^3 - x_k^3 y_l^2) t^1, \\ t_{kl} = 2(y_k^2 y_l^3 t^1 + y_k^3 y_l^1 t^2 + y_k^1 y_l^2 t^3).$$

In the simplest case, where $F_k(Q, P) = P_k$, the commutator of two elements z_1 and z_2 can be written as

$$[z_1, z_2] = \imath z_3, \quad (49)$$

where

$$s_3 = x_k^1 y_k^2 - x_k^2 y_k^1, \quad x_k^3 = t^3 = 0, \quad y_k^3 = t^1 y_k^2 - t^2 y_k^1, \quad (50)$$

and the Jacobian of three arbitrary elements z_1, z_2 , and z_3 has the form

$$J[z_1, z_2, z_3] = z_4, \quad (51)$$

where

$$s_4 = -s_{kl} \delta_{kl}, \quad x_k^4 = y_k^4 = t^4 = 0.$$

4. Conclusion

It is known that one can describe a dissipative system starting from a closed system if the dissipative system in question is a part of the closed system. However, situations can arise where it is difficult or impossible to find a closed system comprising the given dissipative system. For instance, this is the case for the quantization of an electromagnetic field in a resonator with losses [2, 9].

Note that models of quantum dissipative systems can play a more important part in fundamental theories than they have been assumed to up to now. For instance, the importance of quantum dissipative systems in the theory of strings and superstrings stems, among other things, from the following.

1. In spaces of noncritical dimension (e.g., in four-dimensional space), strings are dissipative systems in the phase space of the "coupling constants." Here the dissipative force is determined by the nonzero beta-functions of the coupling constants [10].

2. Decay of the pure quantum state to a mixed state may occur on the string level due to quantum fluctuations of the metrics, which are virtual black holes on a two-dimensional surface covered by a string in the course of its motion. The result is the necessary nonunitary generalization of the von Neumann equation [7].

3. The energy and momentum conservation laws normally stem from an a priori limitation on the properties of space-time geometry. However, it would be more desirable and consistent to obtain the limitations on the geometry within the framework of a more general theory than to merely postulate the geometry. For example, deriving limitations on the properties and structures of space-time from the requirement of the self-consistency of quantum theory is analogous to deducing the gauge group and space-time dimension in theories of strings and superstrings.

4. In a wide range of non-Riemannian geometries, e.g., in the curved affine-metric space-time, the closed bosonic string is a dissipative system [11-13].

In the present paper, we have shown that the requirement for a dynamic description of dissipative (non-Hamiltonian) systems to be self-consistent forces one to reject Jacobi identities for the system evolution generators and to use anticommutative non-Lie algebras instead. The properties of these algebras will be described in Part IV of this paper.

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Appendix

Leibnitz rule and dissipative systems

The inconsistency of the quantum equations for the evolution of a dissipative system and the quantum commutation relations is evident when one examines the time derivatives of the commutators of the coordinates and momenta using the Leibnitz rule and the Jacobi identity. This inconsistency stems from the assumption that the Leibnitz rules and Jacobi identities hold for dissipative systems and is due to the fact that the equations of evolution in time for dissipative systems violate the structure of the Lie algebra.

Note that for a quantum Hamiltonian system, the rule of term-by-term differentiation with respect to time (Leibnitz rule) $D(AB) = D(A)B + AD(B)$ is valid for the product of any operators. The Leibnitz rule does not, however, apply to quantum dissipative systems.

Statement A.1. *Application of the rule of term-by-term differentiation with respect to time (Leibnitz rule) to the quantum commutation relations of coordinate and momentum operators describing the quantum dissipative system (13) necessarily entails the fulfillment of conditions (15)-(17) for the right-hand sides of Eqs. (13) and, thus, equivalence of the motion equations (12) to the evolution equations for Hamiltonian system (11).*

Proof. The quantum commutation relations for the coordinate and momentum operators have the form

$$[Q_k(t), P_l(t)] = i\hbar\delta_{kl}I, \quad [Q_k(t), Q_l(t)] = [P_k(t), P_l(t)] = 0. \quad (52)$$

Considering the time derivative from the first relation in (52), we obtain

$$\frac{d}{dt}[Q_k(t), P_l(t)] = 0. \quad (53)$$

In this case, the Leibnitz rule has the form

$$\frac{d}{dt}[Q_k(t), P_l(t)] = \left[\frac{d}{dt}Q_k(t), P_l(t) \right] + \left[Q_k(t), \frac{d}{dt}P_l(t) \right]. \quad (54)$$

Consequently, we have the relation

$$\left[\frac{d}{dt}Q_k(t), P_l(t) \right] + \left[Q_k(t), \frac{d}{dt}P_l(t) \right] = 0. \quad (55)$$

Using motion equation (13), we obtain

$$\frac{i}{\hbar} \left([[H, Q_k], P_l] + [Q_k, [H, P_l]] \right) + ([Q_k, F_l] - [G_k, P_l]) = 0. \quad (56)$$

The Jacobi identity for the associative operators Q_k, P_l, H has the form

$$J[Q_k, P_l, H] = \frac{1}{\hbar^2} \left([[Q_k, P_l], H] + [[P_l, H], Q_k] + [[H, Q_k], P_l] \right) = 0,$$

i.e.,

$$[[H, Q_k], P_l] + [Q_k, [H, P_l]] = 0.$$

As a result, we obtain the identity

$$\Omega^2_{kl} = -\frac{i}{\hbar} ([G_k, P_l] - [Q_k, F_l]) = 0. \quad (57)$$

Similarly, considering other quantum commutation relations (52), one can obtain all of the identities (15)-(17).

Thus, the statement that the Leibnitz rule is valid for quantum dissipative systems (13) is equivalent to the requirement that conditions (15)-(17) be fulfilled, i.e., fulfillment of the conditions for the Hamiltonian (nondissipative) character of system [14].

Thus, the rule of term-by-term differentiation with respect to time (Leibnitz rule) does not apply to quantum dissipative systems. This is the essential feature distinguishing dissipative and Hamiltonian systems. That the time derivative operator and the evolution operator do not obey the Leibnitz rule means, mathematically, that these operators are not differentiation operators. They are usually called dissipative operators and we give their definitions below.

Definition A.1. The operator D is called a *dissipative operator* if it obeys the generalized Leibnitz rule

$$D(AB) = D(A)B + AD(B) + Z(A, B)$$

and meets the conditions

$$D(A + B) = D(A) + D(B)$$

and

$$D(\lambda A) = \lambda D(A), \quad \lambda \in R;$$

$Z(A, B)$ is a nonzero operator depending on the form of operators A and B .

Thus, the dissipation operator that describes evolution (12) of a quantum dissipative system and, consequently, the time derivative operator, are dissipative operators.

Note that violation of the Leibnitz rule and the appearance of dissipative operators as evolution operators are well known in the theory of quantum dynamic semigroups [15] used to describe open systems.

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