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QUANTUM DISSIPATIVE SYSTEMS. IV. ANALOGUES OF LIE ALGEBRAS AND GROUPS

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The condition of self-consistency for the quantum description of dissipative systems makes it necessary to exceed the limits of Lie algebras and groups, i.e., this requires the application of non-Lie algebras (in which the Jacobi identity does not hold) and analytic quasi-groups (which are nonassociative generalizations of groups). In the present paper, we show that the analogues of Lie algebras and groups for quantum dissipative systems are commutant-Lie algebras (anticommutative algebras whose commutants are Lie subalgebras) and analytic commutant-associative loops (whose commutants are associative subloops (groups)). It is proved that the tangent algebra of an analytic commutant-associative loop with invertibility (a Valya loop) is a commutant-Lie algebra (a Valya algebra). Some examples of commutant-Lie algebras are considered.

1. Introduction

When describing nondissipative (Hamiltonian) systems in quantum mechanics, one often makes use of a pair consisting of a Lie algebra and an analytic group (a Lie group). The relationship between the members of the pair is described by the Lie theorems [1, 2]. A noncontradictory description of the quantum evolution of dissipative (non-Hamiltonian) systems needs to exceed the limits of this pair [3]. Consequently, it is necessary to resort to an anticommutative non-Lie algebra (in which the Jacobi identity does not hold) and an analytic nonassociative quasi-group, i.e., a loop [2, 4].

Unfortunately, the algebras and loops required for a quantum description of dissipative systems have not been thoroughly studied [5–7]. In this fourth part of the present work, we prove that the desired pair is an algebra whose commutant is a Lie subalgebra (a Valya algebra) and an analytic loop, with an associative subloop (group) as its commutant. An analogue of the Lie theorem relating this analytic loop to a Valya algebra is proved. Some examples of non-Lie algebras with Lie commutants are presented, including the generalized Heisenberg–Weyl algebra suggested in [3] for describing quantum dissipative systems, a generalization of the Poisson algebra for differential 1-forms and vector fields, and also an algebra of infinite matrices whose commutants are associative matrices.

2. Commutant-Lie algebras

2.1. The Valya algebra, commutant, and commutant-Lie algebra. We state the following definition:

Definition 2.1. An algebra B is called a **Valya algebra** if the operation of multiplication satisfies

(1) the anticommutativity condition $xx = 0$, and

(2) the mild Jacobi identity $J(xy, zp, ql) = 0$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements of the algebra.

We can see that any Lie algebra defined by the relations $x^2 = 0$ and $J(x, y, z) = 0$ is a Valya algebra. Every binary Lie algebra [2], i.e., an algebra whose operation of multiplication satisfies the conditions $x^2 = 0$ and $J(x, y, xy) = 0$, is also a Valya algebra.

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Let A be a nonassociative algebra. We introduce the bilinear operation $[x, y] = xy - yx$ called a commutator. As a result, we obtain the commutator algebra $A^{(-)}$ associated with algebra A. The anticommutativity condition and the mild Jacobi identity are realized for the commutator algebra in the form

(1) $[x, x] = 0$,

(2) $J\big[[x,y],[z,p],[q,l]\big] = 0$, where $J[x,y,z] = \big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big]$.

Furthermore, let $g = A^{(-)}$ be the commutator algebra associated with an algebra A. The commutator algebra admits the definition of the concept of a commutant [2].

Definition 2.2. The **commutant** of algebra g is the subspace $[g, g] = g'$ generated by all commutators $[x, y]$, where $x, y \in g$.

Definition 2.3. The **multiple commutants** $g^{(k)}$, $k = 0, 1, 2, \ldots$, of the algebra g are defined by the following induction rule: $g^{(0)} = g$, $g^{(k+1)} = (g^{(k)})'$.

Recall that by the **ideal** A_0 of an algebra A is meant a subalgebra of algebra A such that the commutators of the elements of the subalgebra with an arbitrary element of the algebra are elements of the subalgebra. It is easy to prove the following assertion.

Proposition 2.1.

1. *The commutant $g^{(1)}$ of the algebra g is an ideal, whereas $g^{(2)}$ is not an ideal of the algebra g.*
2. *The multiple commutant $g^{(k+1)}$ is an ideal of the algebra $g^{(k)}$, but is not an ideal of the algebra $g^{(l)}$, where $l < k$.*
3. *The multiple commutant $g^{(k+1)}$ is an ideal of the algebra $g^{(l)}$, where $l < k$, only if the Jacobi identity holds.*

We introduce the next definition.

Definition 2.4. Let $A^{(-)}$ be the commutator algebra associated with an algebra A. We say that $A^{(-)}$ is a **commutant-Lie algebra** if any three of its commutants generate a subalgebra that is a Lie algebra, i.e., if the commutant of this algebra is a Lie subalgebra.

It is quite clear that a commutant-Lie algebra is a Valya algebra.

2.2. The generalized Heisenberg–Weyl algebra and the Valya algebra. We consider some properties that the generalized Heisenberg–Weyl algebra W_N^*, defined in the third part of [3], must possess.

Theorem 2.1. *The generalized Heisenberg–Weyl algebra W_N^* is a Valya algebra, but is not a Lie algebra, i.e., W_N^* is a non-Lie Valya algebra.*

Proof. Using the relation

$$[z_n, z_m] = \imath s_3^{nm} I + \imath t_k^{nm} F_k(Q, P) \tag{1}$$

for the commutators of the elements z_n and z_m of the generalized Heisenberg–Weyl algebra, where

$$s_3^{nm} = x_k^n y_k^m - x_k^m y_k^n, \qquad t_k^{nm} = t^n y_k^m - t^m y_k^n,$$

we obtain the expression

$$J\big[[z_1, z_2], [z_3, z_4], [z_5, z_6]\big] = -i t_k^{12} t_l^{34} t_m^{56} J[F_k, F_l, F_m]$$

for the Jacobian of the commutators. Because $F_k(Q, P)$ are associative operators, we derive the relation

$$J[F_k, F_l, F_m] = 0.$$

Hence, the elements of the generalized Heisenberg–Weyl algebra [3] satisfy the mild Jacobi identity.

It can be shown that the generalized Heisenberg–Weyl algebra W_N^* and its subalgebras possess the following properties.

1. The Heisenberg–Weyl algebra W_N is a two-sided ideal of the generalized Heisenberg–Weyl algebra W_N^* and, by virtue of the identity $[W, Q_i] = 0$, this two-sided ideal is not a completely prime ideal.

2. The commutant of the generalized Heisenberg–Weyl algebra is a subalgebra of the Heisenberg–Weyl algebra.

3. The maximal two-sided ideal of the generalized Heisenberg–Weyl algebra W_N^*—this ideal is a Lie algebra—coincides with the Heisenberg–Weyl algebra W_N.

Consider the simplest example of a commutant-Lie generalization of the Heisenberg–Weyl algebra W_N, namely, the linear generalized Heisenberg–Weyl algebra LW_N^* defined in [3]. We can now state a proposition.

Proposition 2.2. *The generalized Heisenberg–Weyl algebra LW_N^* satisfies the following stronger Jacobi identities in addition to the mild Jacobi identity*:

$$J\big[[z_1, z_2], [z_3, z_4], z_5\big] = 0, \tag{2}$$

$$\big[J[z_1, z_2, z_3], z_4\big] = 0, \tag{3}$$

$$\big[[z_1, z_2], [z_3, z_4]\big] = 0. \tag{4}$$

Note that relation (2) *and the mild Jacobi identity it implies are a consequence of identity* (4).

To prove the identities (2)–(4), it suffices to apply relation (1) and the definition in [3] of the algebra LW_N^*. The commutator algebra defined by relation (4) is called a **commutant-commutative algebra**. This is a solvable algebra [2] because there exists a multiple commutant equal to the zero element of the algebra and, consequently, the chain of multiple commutants terminates.

As a result, the linear generalized Heisenberg–Weyl algebra LW^* is a commutant-commutative algebra whose Jacobians belong to the center of the algebra. The solvability of this algebra permits the idea of a radical, i.e., the greatest solvable ideal, to be introduced, which simplifies the study of the structural properties of the algebra.

2.3. Commutant-associative algebra. Recall that an algebra is called an **associative algebra** if the associator of its three arbitrary elements is zero, i.e., the operation of multiplication satisfies the condition $(x, y, z) = 0$, where $(x, y, z) \equiv (xy)z - x(yz)$ is the associator of the operators x, y, and z. We now state the following definitions.

Definition 2.5. A nonassociative algebra is called a **commutant-associative algebra** (or, simply, a **commutant algebra**) if the operation of multiplication satisfies the condition

$$(xy - yx, zp - pz, ql - lq) = 0. \tag{5}$$

Definition 2.6. A nonassociative algebra is called a **quasi-commutant algebra** if the operation of multiplication satisfies both

(1) the left quasi-commutant condition $(xy - yx, zp - pz, q) = 0$, and

(2) the right quasi-commutant condition $(q, xy - yx, zp - pz) = 0$.

As can be easily seen, a quasi-commutant algebra is commutant-associative. The converse is false. In addition, the commutator algebra of an arbitrary quasi-commutant algebra satisfies identity (2). We state the following theorem.

Theorem 2.2. *The commutator algebra $A^{(-)}$ of an arbitrary commutant-associative algebra A is a commutant-Lie algebra.*

Proof. The proof of the theorem is based on the property that the algebraic Jacobian defined for the commutator algebra by the formula

$$J[A, B, C] = \big[[A, B], C\big] + \big[[B, C], A\big] + \big[[C, A], B\big] \tag{6}$$

can be written in terms of associators as

$$J[A,B,C] = (A,B,C) - (A,C,B) + (C,A,B) - (B,A,C) + (B,C,A) - (C,B,A).$$

Consequently, the desired assertion can be proved using Definition 2.4.

2.4. Generalized Heisenberg–Weyl algebra and its associator. As was shown in [8], the generalized Heisenberg–Weyl algebra can be realized as the commutator algebra of a nonassociative algebra.

Denote by $E_k = (Q_k, P_k)$ the base elements of the Heisenberg–Weyl algebra. Then, for the commutation relations of the generalized Heisenberg–Weyl algebra [3] to hold, it suffices that the base elements (E_k, W) satisfy the relations

$$(E_k, E_l, E_j) = (E_k, E_l, W) = (W, E_k, E_l) = (E_k, W, E_k) = (W, W, W) = 0, \tag{7}$$

$$(E_k, W, E_l) \not\equiv 0, \qquad (W, W, E_k) \not\equiv 0, \qquad (E_k, W, W) \not\equiv 0, \qquad (W, E_k, W) \not\equiv 0. \tag{8}$$

Using (7) and (8), it is easy to prove the following assertion.

Proposition 2.3. *A nonassociative algebra whose elements can be represented in the form*

$$z = x_k E_k + tW,$$

where x_k and t are numbers and the base elements (E_k, W) satisfy relations (7) *and* (8), *is a quasi-commutator algebra, i.e.,*

$$(z_1 z_2 - z_2 z_1, z_3 z_4 - z_4 z_3, z_5) = (z_5, z_1 z_2 - z_2 z_1, z_3 z_4 - z_4 z_3) = 0.$$

Moreover, by virtue of relations (1), the base elements (E_k, W) satisfy the commutant-associativity relations (5).

3. Loops with associative commutants

3.1. Loops with invertibility and loop commutants. By analytic loops we mean those nonassociative generalizations of analytic groups (Lie groups) that were first considered by Mal'tsev [5].

A set G with a binary multiplication operation $\circ$ is called a **quasi-group** if, for any $a, b \in G$, each of the equations $a \circ x = b$ and $y \circ a = b$ has a unique solution. If a quasi-group G contains an (identity) element $e \in G$ such that $e \circ x = x \circ e = x$ for all $x \in G$, then the quasi-group is called a loop [4].

Let G be a **loop**, i.e., a set with operations of multiplication and right and left division ($\circ$, $/$, and $\backslash$) that contains an element $e \in G$ such that $x \circ e = e \circ x = x$ and $(x/y) \circ y = y \circ (y \backslash x) = (x \circ y)/y = y \backslash (y \circ x) = x$ for all x and y belonging to G.

Although a loop possesses an identity element e, though because it is a quasi-group, it contains the elements e/a and $e \backslash a$ for any element a, these elements are not inverse elements in an arbitrary loop. Therefore, we consider a narrower class of loops. A loop G is called a **loop with invertibility** if any two of its elements $a, b \in G$ satisfy the relations

$$(b \circ a) \circ (e/a) = b, \qquad (e \backslash a) \circ (a \circ b) = b. \tag{9}$$

The substitution of the element e/a for b in the first of these relations results in $e/a = e \backslash a$. Thus, every element of a loop with invertibility possesses a unique two-sided inverse element a^{-1}, i.e.,

$$a \circ a^{-1} = a^{-1} \circ a = e. \tag{10}$$

Relations (9) defining a loop with invertibility now acquire the form

$$(b \circ a) \circ a^{-1} = a^{-1} \circ (a \circ b) = b. \tag{11}$$

We note that for the loops with invertibility, the following very important identity holds:

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1}. \tag{12}$$

Indeed, if $a \circ b = c$, then we have

$$b = a^{-1} \circ c \Rightarrow a^{-1} = b \circ c^{-1} \Rightarrow c^{-1} = b^{-1} \circ a^{-1}.$$

It is possible to define the concepts of commutator and commutant for a loop.

By the **commutator** of two elements x and y of a loop G, we mean an element z of the form

$$z = [x, y] = (x \circ y) \circ (y \circ x)^{-1}. \tag{13}$$

In view of identity (12), the commutator of two elements x and y in a loop with invertibility can be represented as $[x, y] = (x \circ y) \circ (x^{-1} \circ y^{-1})$.

The **commutant of a loop** G is defined as the set $G^{(-)}$ of all elements $z \in G$ of the loop G that are representable in the form $z = z_1 \circ z_2 \circ \cdots \circ z_m$, where each element z_i is the commutator of two elements $x_i, y_i \in G$, i.e., $z_i = (x_i \circ y_i) \circ (y_i \circ x_i)^{-1}$.

3.2. Tangent algebra of an analytic loop. An **analytic loop** is an analytic manifold endowed with the structure of a loop such that the operation of multiplication is analytic [5, 2].

By definition, the **tangent algebra** g of a local analytic loop G is the tangent space $T_e(G)$ in which the binary and ternary operations $[\,,\,]$ and $\langle\,,\,,\,\rangle$ are defined in the following way. Let $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ be differentiable paths in the loop G that satisfy the conditions $\alpha(0) = \beta(0) = \gamma(0) = e$, $\alpha'(0) = \xi$, $\beta'(0) = \eta$, and $\gamma'(0) = \zeta$. Then,

$$(\beta(t) \circ \alpha(t)) \backslash (\alpha(t) \circ \beta(t)) = t^2[\xi, \eta] + o(t^2), \tag{14}$$

$$(\alpha(t) \circ (\beta(t) \circ \gamma(t))) \backslash ((\alpha(t) \circ \beta(t)) \circ \gamma(t)) = t^3\langle \xi, \eta, \zeta \rangle + o(t^3). \tag{15}$$

In a local coordinate system on an analytic loop G, in the neighborhood of the identity element, the product of two elements $z_i = \mu_i(x, y)$ can be expanded into a Taylor series as

$$\mu_i(x, y) = x_i + y_i + a^i_{jk} x_j y_k + b^i_{jkl} x_j x_k y_l + c^i_{jkl} x_j y_k y_l + \cdots. \tag{16}$$

The loop contains an identity element and, therefore, as in the case of a Lie group, the functions $\mu_i(x, y)$ possess the property

$$\left[\frac{\partial^r \mu_i}{\partial x_{i_1} \ldots \partial x_{i_r}}\right]_{x=0, y=0} = 0, \qquad \left[\frac{\partial^r \mu_i}{\partial y_{i_1} \ldots \partial y_{i_r}}\right]_{x=0, y=0} = 0. \tag{17}$$

In a local coordinate system, the binary and ternary operations have the form

$$[\xi, \eta]_i = u^i_{jk} \xi^j \eta^k, \qquad \langle \xi, \eta, \zeta \rangle_i = v^i_{jkl} \xi^j \eta^k \zeta^l, \tag{18}$$

where

$$u^i_{jk} = a^i_{jk} - a^i_{kj}, \qquad v^i_{jkl} = 2b^i_{jkl} - 2c^i_{jkl} + \frac{1}{4} u^i_{ml} v^m_{jk} - \frac{1}{4} v^i_{jm} v^m_{kl}. \tag{19}$$

If the local loop is associative, i.e., is a Lie group, then the ternary operation is zero, i.e., $\langle \xi, \eta, \zeta \rangle = 0$ for all $\xi, \eta, \zeta \in g$. The ternary operation in the tangent algebra for a diassociative (alternative) local analytic loop (a loop in which any pair of elements generates an associative subloop) can be expressed in terms of the binary operation by the formula

$$\langle \xi, \eta, \zeta \rangle = -\frac{1}{6} J[\xi, \eta, \zeta],$$

where

$$J[\xi,\eta,\zeta] = [[\xi,\eta],\zeta] + [[\eta,\zeta],\xi] + [[\zeta,\xi],\eta].$$

Thus, for an arbitrary loop, there is a formula in canonical coordinates that relates the operation of multiplication in the loop to that of the tangent algebra, namely, we have $x \circ y = x + y + (1/2)xy + \cdots$, where xy is the binary anticommutative operation of multiplication (i.e., $xx = 0$ or, which is the same, $xy = -yx$) of the corresponding elements of the tangent algebra of the given loop.

Let us show that the commutator of the loop is expressed in canonical coordinates by the formula

$$[x,y] \equiv (x \circ y) \circ (y \circ x)^{-1} = xy + \cdots, \tag{20}$$

where the dots symbolize elements of the third or higher degree. Indeed, applying the relations

$$x \circ y = x + y + (1/2)xy + \cdots, \qquad (y \circ x)^{-1} = -y - x - (1/2)yx - \cdots,$$

we obtain the expression

$$[x,y] = (x \circ y) \circ (y \circ x)^{-1} = (1/2)(xy - yx + xx - xy - yx + yy) + \cdots$$

for the commutator of two elements of the loop. If the binary operation in the tangent algebra is anticommutative (i.e., $xx = 0$ and $xy = -yx$), we arrive at the desired relation (20).

3.3. Tangent algebra of a commutant-associative loop. We state the following definition.

Definition 3.1. By a **commutant-associative loop (a Valya loop)** we mean a loop with invertibility whose commutant is an associative subloop (i.e., a group).

The classical correspondence between analytic local groups (Lie groups) and Lie algebras, which is established by the Lie theorems [1], also exists between analytic local commutant-associative loops with invertibility (Valya loops) and commutant-Lie algebras (Valya algebras).

Theorem 3.1.

A. *The tangent algebra of an analytic local commutant-associative loop is a commutant-Lie algebra.*

B. *Every finite commutant-Lie algebra is the tangent algebra of a analytic local commutant-associative loop that is unique (to within the local isomorphisms).*

Proof.

A. Consider an arbitrary element of the commutant of a loop G with invertibility. By definition of the commutant, it is representable in the form $z_1 \circ z_2 \circ \cdots \circ z_m$, where each element z_k is the commutator of two elements x_k and y_k of the loop G. If G is a commutant-associative loop, then for small values of t_k, $k = 1,2,\ldots,m$, the product of the form $g_1(t_1) \circ g_2(t_2) \circ \cdots \circ g_m(t_m)$, where $g_k(t_k)$ are arbitrary one-parameter subgroups with tangent vectors z_k, does not depend on the arrangement of the brackets. Consequently, the subalgebra generated by the elements z_k of the tangent algebra is a Lie algebra, i.e., the tangent algebra itself is a commutant-Lie algebra.

B. We introduce coordinates of the first kind. Note that the one-parameter subgroups $g_k(t_k)$ generate an associative local subloop. It can be seen that expansion into a Taylor series in canonical coordinates takes place in the loop, which makes it possible to reconstruct the loop from the algebra.

The proof of assertion B in the general case is, at present, an open question.

4. Representing a commutant-Lie algebra as an algebra of differential forms

4.1. Poisson algebra for differential 1-forms. As is known, a **symplectic manifold** (M^{2n},ω) is defined as a differentiable manifold M^{2n} of even dimension $2n$ on which a differential form ω of order 2 (closed ($d\omega = 0$) and nondegenerate) is defined.

For closed differential 1-forms, we can define the Poisson bracket [9, 10] on a symplectic manifold. The **Poisson bracket for two closed differential 1-forms** $\alpha = a_k(z)dz^k$ and $\beta = b_k(z)dz^k$ on the symplectic manifold (M, ω) is an exact 1-form (α, β) defined according to the rule

$$(\alpha, \beta) = d\Psi(\alpha, \beta) = d\omega(X_\alpha, X_\beta), \tag{21}$$

where

$$\Psi(\alpha, \beta) = \omega(X_\alpha, X_\beta) = \Psi^{kl} a_k b_l, \qquad d\alpha = d\beta = 0, \tag{22}$$

and X_α is a vector field that is related to the corresponding 1-form α by the formula $i(X_\alpha)\omega = i_{X_\alpha}\omega = \alpha$, ω is a closed nondegenerate 2-form [9, 10] called the symplectic form, i symbolizes inner multiplication of the vector fields and differential forms [9], Ψ is the cosymplectic structure, and Ψ^{kl} is a matrix which is the inverse of the matrix of the symplectic form and satisfies the following conditions [10]:

(a) it is antisymmetric, i.e., $\Psi^{kl} = -\Psi^{lk}$;

(b) the Schouten brackets vanish as

$$[\Psi, \Psi]^{slk} = \Psi^{sm}\partial_m\Psi^{lk} + \Psi^{lm}\partial_m\Psi^{ks} + \Psi^{km}\partial_m\Psi^{sl} = 0.$$

If the bilinear operation "Poisson bracket" is defined on the space of the closed 1-forms $\Lambda^1_0(M)$, then M is called a Poisson manifold and the space $\Lambda^1_0(M)$ with this bilinear operation is called a Poisson algebra and is denoted by P_1. The Poisson algebra P_1 is a Lie algebra. This follows from the fulfillment of the anticommutativity condition $(\alpha, \beta) = -(\beta, \alpha)$ and the Jacobi identity

$$J(\alpha, \beta, \gamma) = ((\alpha, \beta), \gamma) + ((\beta, \gamma)\alpha) + ((\gamma, \alpha), \beta) = 0. \tag{23}$$

If definition (21) is extended to arbitrary 1-forms, including nonclosed ones, then the Jacobi identity does not hold for the nonclosed 1-forms: $J(\alpha, \beta, \gamma) \not\equiv 0$. Therefore, the generalization of the Poisson algebra, i.e., the space of differential 1-forms $\Lambda^1(M)$ with this bilinear operation, is not a Lie algebra.

Some authors [10, 9] considered the generalization of the Poisson bracket to nonclosed differential forms satisfying the Jacobi identity. In this case, the Poisson algebra of nonclosed differential 1-forms is a Lie algebra. For instance, the **Poisson bracket of two differential 1-forms** $\alpha = a_k(z)dz^k$ and $\beta = b_k(z)dz^k$ on a symplectic manifold (M, ω) can be introduced as the 1-form (α, β) defined according to the rule [10]

$$(\alpha, \beta) = d\Psi(\alpha, \beta) + \Psi(d\alpha, \beta) + \Psi(\alpha, d\beta). \tag{24}$$

This Poisson bracket satisfies the anticommutativity condition and the Jacobi identity (23). Therefore, the Poisson algebra of nonclosed differential forms with Poisson brackets of type (24) is a Lie algebra.

4.2. Nonclosed forms and dissipative systems. We note that the non-Hamiltonian character and dissipative properties of dynamic systems are related to the properties of nonclosed differential forms. Following [9], we define a **Hamiltonian system** on a symplectic manifold (M^{2n}, ω) as a vector field X on M^{2n} such that the differential 1-form $i_X\omega$ is closed. If the form $i_X\omega$ is exact, then there exists a differentiable function H on M^{2n}, which is called the **Hamiltonian** of the system, such that $i_X\omega = -dH$.

As is known, the system determined by a vector field X on a symplectic manifold (M^{2n}, ω) is a Hamiltonian system if and only if the Lie derivative of the symplectic form with respect to the vector field vanishes, i.e., $L_X\omega = 0$, where $L_X = di_X + i_Xd$ is the Lie derivative with respect to the vector field X.

We state the following obvious definition.

Definition 4.1. By a **dissipative (non-Hamiltonian) system** on a symplectic manifold (M^{2n}, ω) we mean a vector field X on M^{2n} such that the differential 1-form $i_X\omega$ is nonclosed.

Thus, the construction of the commutant-Lie algebra of nonclosed differential forms and the corresponding vector fields makes it possible to describe the properties of classical dissipative systems on a symplectic manifold.

4.3. Generalized Poisson algebra for differential forms. In order to describe classical dissipative systems, one suggestion is to generalize the Poisson algebra P_1. We define a binary operation on $\Lambda^1(M)$.

Definition 4.2. The **generalized Poisson bracket** for two 1-forms α and β is the exact differential form $[\alpha, \beta]$ defined according to the rule

$$[\alpha, \beta] = d\Psi(\alpha, \beta) = d\omega(X_\alpha, X_\beta). \tag{25}$$

This definition results from direct generalization of the Poisson bracket (21) to nonclosed forms. The Poisson bracket (25) for nonclosed 1-forms does not satisfy the Jacobi identity, i.e.,

$$J[\alpha, \beta, \gamma] = \big[[\alpha, \beta], \gamma\big] + \big[[\beta, \gamma], \alpha\big] + \big[[\gamma, \alpha], \beta\big] \not\equiv 0.$$

Therefore, the generalized Poisson algebra P_1^* is not a Lie algebra. The Jacobi identity does hold for closed 1-forms. Hence, the closed 1-forms determine a Lie algebra that is the Poisson algebra P_1. As a result, the generalized Poisson algebra P_1^* contains a subalgebra that is a Poisson algebra. The generalized Poisson bracket of two nonclosed 1-forms is a closed 1-form and, therefore, the subalgebra P_1 is a two-sided ideal of the algebra P_1^*. Consequently, there exists an exact diagram of the form

$$0 \to P_1 \to P_1^* \to P_1^*/P_1 \to 0.$$

The generalized Poisson bracket for 1-forms possesses the following properties:

(1) it is anticommutative, i.e., $\forall \alpha, \beta \in P_1^* \quad [\alpha, \beta] = -[\beta, \alpha] \in P_1$;

(2) satisfies the Jacobi identity $\forall \alpha, \beta, \gamma \in P_1 \quad J[\alpha, \beta, \gamma] = 0$;

(3) is non-Lie, i.e., $\forall \alpha, \beta, \gamma \in P_1^* \quad J[\alpha, \beta, \gamma] \not\equiv 0$;

(4) satisfies the distributivity condition $\forall \alpha, \beta, \gamma \in P_1^* \quad [a\alpha + b\beta, \gamma] = a[\alpha, \gamma] + b[\beta, \gamma]$, where a and b are real numbers.

The following theorem can be easily proved.

Theorem 4.1. *The generalized Poisson algebra is a commutant-Lie algebra, i.e., the generalized Poisson bracket* (25) *satisfies the anticommutativity condition* $[\alpha, \alpha] = 0$ *and the mild Jacobi identity*

$$J\big[[\alpha, \beta], [\gamma, \delta], [\mu, \nu]\big] = 0. \tag{26}$$

Thus, the structure of a commutant-Lie algebra that is not a Lie algebra can be defined in a natural way on the space of differential 1-forms $\Lambda^1(M)$ on the symplectic manifold M. A commutant-Lie algebra of differential forms is a generalization of the Lie algebra of closed forms and it contains the given Lie algebra as its subalgebra (ideal).

4.4. Generalized Poisson algebra for vector fields. By analogy with the Poisson algebra of differential forms, the Poisson algebra can be generalized for vector fields. We state another definition.

Definition 4.3. By the **generalized Poisson bracket** for two vector fields X and Y, we mean the vector field $\{X, Y\}$ defined according to the formula

$$\{X, Y\} = Z_{d\omega(X,Y)}, \tag{27}$$

where $d\omega(X, Y) = d\Psi(i_X\omega, i_Y\omega)$ and Z_α is the vector field that corresponds to the differential 1-form α according to the following rule: $i_{Z_\alpha}\omega = \alpha$.

It is known [9] that if two vector fields X and Y have corresponding closed differential forms, i.e., $d(i_X\omega) = d(i_X\omega) = 0$, then the generalized Poisson bracket for these vector fields coincides with their commutator, i.e., $\{X, Y\} = -[X, Y]$. To prove this assertion, it is necessary to apply the identities

$$Z_{i_X\omega} = X \quad \text{and} \quad d\omega(X, Y) = -i_{[X,Y]}\omega - i_Y d(i_X\omega) + i_X d(i_Y\omega), \tag{28}$$

where $[X,Y] = XY - YX$ is the commutator. This permits the relation

$$\{X,Y\} = Z_{d\omega(X,Y)} = Z_{i_{[X,Y]}}\omega = -[X,Y] \tag{29}$$

to be easily derived.

In the general case, the relationship between the generalized Poisson brackets and the commutators of vector fields has the form

$$\{X,Y\} = -[X,Y] - Z_{\beta(X,Y)}, \tag{30}$$

where $\beta(X,Y) = i_Y d(i_X\omega) - i_X d(i_Y\omega)$. We can see that the Jacobi identity does not hold for the generalized Poisson brackets of vector fields, i.e., $J\{X,Y,Z\} \not\equiv 0$. However, it does hold for the vector fields corresponding to differential 1-forms that are closed. These vector fields determine a Lie algebra, which is a Poisson algebra with the commutator as a bilinear operation. Consequently, we have the following theorem.

Theorem 4.2. *The generalized Poisson algebra for vector fields is a commutant-Lie algebra, i.e., the generalized Poisson bracket* (27) *satisfies the anticommutativity condition* $\{X,Y\} = -\{Y,X\}$ *and the mild Jacobi identity* $J\{\{X,Y\},\{Z,K\},\{L,F\}\} = 0$.

Thus, the structure of a commutant-Lie algebra can be defined in a natural way for the space of the vector fields $T(M)$. This algebra is a generalization of the Lie algebra of vector fields and it contains the given Lie algebra as its subalgebra (ideal). By Definition 4.1, the vector fields corresponding to nonclosed differential forms describe classical dissipative systems. Therefore, the properties of the above commutant-Lie algebra are related to those of classical dissipative systems.

5. Infinite matrices

Here, we present one more example of a commutant-Lie algebra. As is known, the observable quantities in quantum theory can be described using infinite matrices. Infinite matrices in the quantum mechanics of Hamiltonian systems must satisfy the Heisenberg commutation relations. In addition, they must satisfy an additional condition in the form of a Jacobi identity, which results in the associativity of infinite matrices. Therefore, quantum Hamiltonian systems are usually described with the help of the infinite Hilbert matrices. To solve the problem of the quantum description of dissipative systems, one must apply operators for which the Jacobi identity does not hold. This makes it necessary to drop the associativity property for the infinite matrices corresponding to these operators.

It is known that the multiplication of infinite matrices is not associative in the general case, i.e.,

$$(AB)C \not\equiv A(BC) \qquad \text{or} \qquad \sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{ik}b_{kl}\Big)c_{lj} \not\equiv \sum_{k=1}^{\infty} a_{ik}\Big(\sum_{l=1}^{\infty} b_{kl}c_{lj}\Big). \tag{31}$$

This can be easily demonstrated by the following example [11]. Let $a_{kl} = c_{kl} = 1$ for any k and l, and let the following inequality hold:

$$\sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty} b_{kl}\Big) \not\equiv \sum_{k=1}^{\infty}\Big(\sum_{l=1}^{\infty} b_{kl}\Big). \tag{32}$$

Then $(AB)C \not\equiv A(BC)$ because $(AB)C$ and $A(BC)$ are equal to the left- and right-hand sides of (32), respectively. As an example of a matrix satisfying condition (32), we can take the matrix (b_{kl}) with the elements

$$b_{kl} = \frac{(k-l)}{2^{k+l-2}}\frac{(k+l-3)!}{(k-1)(l-1)} \qquad (k>1, \quad l>1),$$

$$b_{k1} = 2^{-(k-1)} \quad (k>1), \qquad b_{1l} = -2^{-(l-1)} \quad (l>1), \qquad b_{11} = 0.$$

In this case, we have

$$\sum_{l=1}^{\infty}\left(\sum_{k=1}^{\infty} b_{kl}\right) = 1, \qquad \sum_{k=1}^{\infty}\left(\sum_{l=1}^{\infty} b_{kl}\right) = -1.$$

Recall that an infinite matrix $A = (a_{pq})$ is a Hilbert matrix if the bilinear form

$$A(x, y) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} x_p a_{pq} y_q,$$

defined as the interior limit on the unit hypersphere, exists [11].

Definition 5.1. A **Hilbert matrix** is defined as an infinite matrix $A = (a_{pq})$ such that for any $\varepsilon > 0$ and an arbitrary pair $x = (x_p)$, $y = (y_q)$ of unit vectors satisfying the condition

$$\sum_{p=1}^{\infty} |x_p|^2 = \sum_{q=1}^{\infty} |y_q|^2 = 1,$$

there exists a number $A(x, y)$ not depending on ε and two numbers $M(\varepsilon)$ and $N(\varepsilon)$ for which

$$\left| A(x, y) - \sum_{p=1}^{m}\sum_{q=1}^{n} x_p a_{pq} y_q \right| < \varepsilon$$

whenever $m > M(\varepsilon)$ and $n > N(\varepsilon)$.

Note that given two arbitrary infinite matrices $A = (a_{pq})$ and $B = (b_{pq})$, the existence of the product AB does not imply the existence of the product BA. For instance, if $a_{ij} = 0 \;\; (j > 1)$, then we have

$$(BA)_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj} = 0 \quad (j > 1), \qquad (BA)_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{k1} \quad (j = 1),$$

whence it follows that BA does not exist if the series $\sum_{k=1}^{\infty} a_{ik} b_{k1}$ are divergent.

We state the next definition.

Definition 5.2. Dissipative matrices are infinite matrices whose products of one another always exist and for which the commutator of two arbitrary matrices $A = (a_{pq})$ and $B = (b_{pq})$ is equal to a Hilbert matrix $H = (h_{pq})$, i.e., $AB - BA = H$.

It is easy to see that, in the general case, dissipative matrices are not associative and do not satisfy the Jacobi identity. However, the following theorem is true.

Theorem 5.1. *Dissipative matrices satisfy the mild Jacobi identity and, consequently, form a commutant-Lie algebra.*

To prove the first assertion of the theorem, it suffices to apply the associativity property of Hilbert matrices and the representation of the Jacobian in terms of the associators we used in Theorem 4.2. Because the sum and the difference of dissipative matrices are dissipative matrices, these matrices obey the distributive law and, consequently, form an algebra. The commutators of these matrices satisfy the anticommutativity condition and the mild Jacobi identity and, therefore, the dissipative matrices form a commutant-Lie algebra.

Note that the distributive law does not apply to arbitrary infinite matrices. This is due to the fact that the existence of the product $A(B + C)$ does not imply the products AB and AC. For example, this is the case for the matrices (a_{ij}), (b_{ij}), and (c_{ij}), where $a_{ij} = 1$, $b_{ij} = d_i + 1$, $c_{ij} = d_i - 1$ for any j, and the series $\sum_{i=1}^{\infty} d_i$ is convergent.

Thus, the algebras of some infinite matrices are not associative, which permits anticommutative non-Lie algebras to be constructed. It appears that in order to describe the evolution of quantum dissipative systems, we must make use of nonassociative infinite matrices whose commutators are associative infinite matrices (e.g., Hilbert matrices). However, the description of the properties of these infinite nonassociative matrices is presently an open question.

6. Conclusion

Here we present the main results of this paper.

1. To describe the dynamics of quantum dissipative (non-Hamiltonian) systems consistently, it is necessary to drop the Jacobi identity and consider anticommutative Lie algebras.
2. An analogue of a Lie algebra for quantum dissipative systems is a commutant-Lie algebra (a Valya algebra), i.e., an anticommutative algebra whose commutators generate a Lie subalgebra (or, in other words, an algebra whose commutant is a Lie algebra).
3. An analogue of a Lie group for quantum dissipative systems is a Valya loop, i.e., a loop whose commutant is an associative subloop (group).

Below we enumerate the examples of the commutant-Lie algebras presented herein.

1. The generalized Heisenberg–Weyl algebra (Theorem 2.1);
2. The commutator algebra associated with a commutant-associative algebra (Definition 2.5 and Theorem 2.2);
3. The tangent algebra of an analytic commutant-associative loop (Definition 3.1 and Theorem 3.1);
4. The generalized Poisson algebra of differential 1-forms and vector fields (Definitions 4.2 and 4.3, and Theorems 4.2 and 4.3);
5. The algebra of infinite dissipative matrices whose commutators are associative matrices (Definition 5.2 and Theorem 5.1).

The above-mentioned algebras are related to classical and quantum non-Hamiltonian (dissipative) systems and, consequently, they are of interest not only from the mathematical standpoint, but also for physics.

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