Theoretical and Mathematical Physics, **158**(3): 355–359 (2009)

FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS FOR ELECTROMAGNETIC WAVES IN DIELECTRIC MEDIA

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We prove that the electromagnetic fields in dielectric media whose susceptibility follows a fractional power-law dependence in a wide frequency range can be described by differential equations with time derivatives of noninteger order. We obtain fractional integro-differential equations for electromagnetic waves in a dielectric. The electromagnetic fields in dielectrics demonstrate a fractional power-law relaxation. The fractional integro-differential equations for electromagnetic waves are common to a wide class of dielectric media regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species (dipoles, electrons, or ions).

Keywords: fractional integro-differentiation, fractional damping, universal response, electromagnetic field, dielectric medium

1. Introduction

Debye formulated his theory of dipole relaxation in dielectrics in 1912 [1]. A large number of dielectric relaxation measurements show that the classical Debye behavior is very rarely observed experimentally [2]–[4]. Dielectric measurements by Jonscher for a wide class of various substances confirm that different dielectric spectra are described by power laws [2], [3].

For the majority of materials, the dielectric susceptibility in a wide frequency range follows a fractional power law, called the universal response [2], [3]. This law is found both in dipolar media beyond their loss peak frequency and in media where the polarization arises from movements of either ionic or electronic hopping charge carriers. It was shown in [5] that the frequency dependence of the dielectric susceptibility $\widetilde{\chi}(\omega) = \chi'(\omega) - i\chi''(\omega)$ follows a common universal pattern for virtually all kinds of media over many decades of frequency,

$$\chi'(\omega) \sim \omega^{n-1}, \qquad \chi''(\omega) \sim \omega^{n-1}, \quad \omega \gg \omega_p, \tag{1}$$

$$\chi'(0) - \chi'(\omega) \sim \omega^m, \qquad \chi''(\omega) \sim \omega^m, \qquad \omega \ll \omega_p, \tag{2}$$

where $\chi'(0)$ is the static polarization, $0 < n, m < 1$, and ω_p is the loss peak frequency. We note that the ratio of the imaginary to the real components of the susceptibility is independent of frequency. The frequency dependence given by (1) implies that the imaginary and real components of the complex susceptibility at high frequencies satisfy the relation

$$\frac{\chi''(\omega)}{\chi'(\omega)} = \coth\frac{\pi n}{2}, \quad \omega \gg \omega_p. \tag{3}$$

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Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 158, No. 3, pp. 419–424, March, 2009. Original article submitted December 18, 2007; revised June 24, 2008.

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Experimental behavior (2) leads to a similar frequency-independent rule for the low-frequency polarization decrement,

$$\frac{\chi''(\omega)}{\chi'(0)-\chi'(\omega)} = \tanh\frac{\pi m}{2}, \quad \omega \ll \omega_p. \tag{4}$$

The laws of universal response for dielectric media [2], [3] can be described using fractional calculus [6]. The theory of integrals and derivatives of noninteger order goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov [6]. Fractional analysis has found many applications in recent studies in mechanics and physics. The interest in fractional equations has been growing continuously during the last few years because of numerous applications. In a short time, the list of applications has become long (see, e.g., [7]–[9]). In [10]–[13], fractional calculus was used to explain the nature of nonexponential relaxation, and equations were obtained containing operators of fractional integration and differentiation.

Here, we prove that a fractional power-law frequency dependence in a time domain gives integro-differential equations with time derivatives and integrals of noninteger order. We obtain fractional equations that describe electromagnetic waves for a wide class of dielectric media. The power laws of Jonscher are represented by fractional integro-differential equations. The electromagnetic fields in the dielectric media demonstrate universal fractional damping. The suggested fractional equations are common (universal) to a wide class of materials regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species.

2. Fractional equations for universal laws

We consider Eqs. (1) and (3). For the region $\omega \gg \omega_p$, universal fractional power law (1) can be represented in the form

$$\widetilde{\chi}(\omega) = \chi_\alpha (i\omega)^{-\alpha}, \quad 0 < \alpha < 1, \tag{5}$$

with some positive constants χ_α and $\alpha = 1-n$. Here,

$$(i\omega)^\alpha = |\omega|^\alpha e^{i\alpha\pi\,\mathrm{sgn}(\omega)/2}.$$

It is easy to see that relation (3) is satisfied for (5).

The polarization density $\mathbf{P}(t,r)$ can be written as

$$\mathbf{P}(t,r) = \mathcal{F}^{-1}\big(\widetilde{\mathbf{P}}(\omega,r)\big) = \varepsilon_0 \mathcal{F}^{-1}\big(\widetilde{\chi}(\omega)\widetilde{\mathbf{E}}(\omega,r)\big), \tag{6}$$

where $\widetilde{\mathbf{P}}(\omega,r)$ is the Fourier transform $\mathcal{F}$ of $\mathbf{P}(t,r)$. Substituting (5) in (6) gives

$$\mathbf{P}(t,r) = \varepsilon_0 \chi_\alpha \mathcal{F}^{-1}\big((i\omega)^{-\alpha}\widetilde{\mathbf{E}}(\omega,r)\big).$$

We note that the Fourier transform of the fractional Liouville integral [6], [14]

$$(I^\alpha_+ f)(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t} \frac{f(t')\,dt'}{(t-t')^{1-\alpha}}$$

is given by the relation (see Theorem 7.1 in [6] and Theorem 2.15 in [14])

$$(\mathcal{F} I^\alpha_+ f)(\omega) = \frac{1}{(i\omega)^\alpha}(\mathcal{F}f)(\omega),$$

where $0 < \operatorname{Re}\alpha < 1$ and $f(t) \in L_1(\mathbb{R})$ or $1 \le p < 1/\operatorname{Re}\alpha$ and $f(t) \in L_p(\mathbb{R})$.

Using the fractional Liouville integral and fractional power law (5) for $\widetilde{\chi}(\omega)$ in the frequency domain, we obtain

$$\mathbf{P}(t,r) = \varepsilon_0 \chi_\alpha (I_+^\alpha \mathbf{E})(t,r), \quad 0 < \alpha < 1. \tag{7}$$

This equation shows that the polarization density $\mathbf{P}(t,r)$ in the high-frequency region is proportional to the fractional Liouville integral of the electric field $\mathbf{E}(t,r)$.

We consider Eqs. (2) and (4). For the region $\omega \ll \omega_p$, universal fractional power law (2) can be represented as

$$\widetilde{\chi}(\omega) = \widetilde{\chi}(0) - \chi_\beta (i\omega)^\beta, \quad 0 < \beta < 1, \tag{8}$$

with some positive constants χ_β, $\widetilde{\chi}(0)$, and $\beta = m$. It is easy to prove that (8) is satisfied for (4).

We note that the Fourier transforms of the fractional Liouville derivative

$$(D_+^\beta f)(t) = \frac{\partial^k}{\partial t^k}(I_+^{k-\beta} f)(t) = \frac{1}{\Gamma(k-\beta)} \frac{\partial^k}{\partial t^k} \int_{-\infty}^{t} \frac{f(t')\,dt'}{(t-t')^{\beta-k+1}},$$

where $k-1 < \beta < k$, are given by the formula (see Theorem 7.1 in [6] and Theorem 2.15 in [14])

$$(\mathcal{F} D_+^\beta f)(\omega) = (i\omega)^\beta (\mathcal{F} f)(\omega),$$

where $0 < \operatorname{Re}\beta < 1$ and $f(t) \in L_1(\mathbb{R})$ or $1 \le p < 1/\operatorname{Re}\beta$ and $f(t) \in L_p(\mathbb{R})$.

Using the definition of the fractional Liouville derivative and fractional power law (8), we can represent polarization density (6) in the form

$$\mathbf{P}(t,r) = \varepsilon_0 \widetilde{\chi}(0) \mathbf{E}(t,r) - \varepsilon_0 \chi_\beta (D_+^\beta \mathbf{E})(t,r), \quad 0 < \beta < 1. \tag{9}$$

This equation shows that the polarization density $\mathbf{P}(t,r)$ in the low-frequency region is determined by the fractional Liouville derivative of the electric field $\mathbf{E}(t,r)$.

Relations (7) and (9) can be considered universal laws. These equations with integro-differentiation of noninteger orders allow obtaining fractional wave equations for the electric and magnetic fields.

3. Universal electromagnetic wave equation

Here, we obtain fractional equations for electromagnetic fields in dielectric media. Using the Maxwell equations, we obtain

$$\varepsilon_0 \frac{\partial^2 \mathbf{E}(t,r)}{\partial t^2} + \frac{\partial^2 \mathbf{P}(t,r)}{\partial t^2} + \frac{1}{\mu}(\operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2 \mathbf{E}) + \frac{\partial \mathbf{j}(t,r)}{\partial t} = 0. \tag{10}$$

For $\omega \gg \omega_p$, the polarization density $\mathbf{P}(t,r)$ is related to $\mathbf{E}(t,r)$ by Eq. (7). Substituting (7) in (10), we obtain the fractional differential equation for the electric field strength

$$\frac{1}{v^2}\frac{\partial^2 \mathbf{E}(t,r)}{\partial t^2} + \frac{\chi_\alpha}{v^2}(D_+^{2-\alpha}\mathbf{E})(t,r) + (\operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2 \mathbf{E}) = -\mu \frac{\partial \mathbf{j}(t,r)}{\partial t}, \tag{11}$$

where $0 < \alpha < 1$ and $v^2 = 1/(\varepsilon_0 \mu)$. We note that $\operatorname{div}\mathbf{E} \neq 0$ for $\rho(t,r) = 0$.

In the region $\omega \ll \omega_p$, the fields $\mathbf{P}(t,r)$ and $\mathbf{E}(t,r)$ are related by Eq. (9). In this case, Eq. (10) becomes

$$\frac{1}{v_\beta^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{a_\beta}{v_\beta^2}(D_+^{2+\beta}\mathbf{E}) + (\operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2 \mathbf{E}) = -\mu \frac{\partial \mathbf{j}}{\partial t}, \quad 0 < \beta < 1, \tag{12}$$

where

$$v_\beta^2 = \frac{1}{\varepsilon_0 \mu [1 + \widetilde{\chi}(0)]}, \qquad a_\beta = \frac{\chi_\beta}{1 + \widetilde{\chi}(0)}.$$

Equations (11) and (12) describe the time evolution of the electric field strength in dielectric media. These equations are fractional differential equations [14] with derivatives of the orders $2 - \alpha$ and $2 + \beta$.

Using the Maxwell equations, we obtain the equation for the magnetic field induction

$$\frac{\partial^2 \mathbf{B}(t,r)}{\partial t^2} = \frac{1}{\varepsilon_0 \mu} \nabla^2 \mathbf{B}(t,r) + \frac{1}{\varepsilon_0} \frac{\partial}{\partial t} \operatorname{curl} \mathbf{P}(t,r) + \frac{1}{\varepsilon_0} \operatorname{curl} \mathbf{j}(t,r). \tag{13}$$

In experiments, the field $\mathbf{B}(t,r)$ can be represented as $\mathbf{B}(t,r) = 0$ for $t \leq 0$ and $\mathbf{B}(t,r) \neq 0$ for $t > 0$. For $\omega \gg \omega_p$, the polarization density $\mathbf{P}(t,r)$ is related to $\mathbf{E}(t,r)$ by Eq. (7), which leads to the fractional differential equation for magnetic field induction in the form

$$\frac{1}{v^2} \frac{\partial^2 \mathbf{B}(t,r)}{\partial t^2} + \frac{\chi_\alpha}{v^2} ({}_0D_t^{2-\alpha} \mathbf{B})(t,r) - \nabla^2 \mathbf{B}(t,r) = \mu \operatorname{curl} \mathbf{j}(t,r), \tag{14}$$

where $0 < \alpha < 1$, $v^2 = 1/(\varepsilon_0 \mu)$, and ${}_0D_t^{2-\alpha}$ is the fractional Riemann–Liouville derivative [14] on $[0, \infty)$ such that

$$({}_0D_t^{2-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \frac{\partial^2}{\partial t^2} \int_0^t \frac{f(t')\, dt'}{(t - t')^{1-\alpha}}, \quad 0 < \alpha < 1.$$

For $\omega \ll \omega_p$, we obtain

$$\frac{1}{v_\beta^2} \frac{\partial^2 \mathbf{B}(t,r)}{\partial t^2} - \frac{a_\beta}{v_\beta^2} ({}_0D_t^{2+\beta} \mathbf{B})(t,r) - \nabla^2 \mathbf{B}(t,r) = \mu \operatorname{curl} \mathbf{j}(t,r), \tag{15}$$

where $0 < \beta < 1$ and

$$v_\beta^2 = \frac{1}{\varepsilon_0 \mu [1 + \widetilde{\chi}(0)]}, \qquad a_\beta = \frac{\chi_\beta}{1 + \widetilde{\chi}(0)}.$$

Equations (14) and (15) are fractional differential equations that describe the magnetic field in dielectric media and demonstrate a power-law relaxation. They can be written in a general form. Such a general fractional differential equation for the magnetic field induction has the form

$$({}_0D_t^{\alpha} \mathbf{B})(t,r) - \lambda_1 ({}_0D_t^{\beta} \mathbf{B})(t,r) - \lambda_2 \nabla^2 \mathbf{B}(t,r) = \mathbf{f}(t,r), \tag{16}$$

where $1 \leq \beta < \alpha < 3$. The curl of the current density of free charges is regarded as an external source: $\mathbf{f}(t,r) = \mu \lambda_2 \operatorname{curl} \mathbf{j}(t,r)$. Equation (16) yields Eq. (14) for $\alpha = 2$, $1 < \beta < 2$, $\lambda_1 = -\chi_\alpha$, and $\lambda_2 = v^2 = 1/(\varepsilon_0 \mu)$. Equation (15) can be written in form (16) for $2 < \alpha < 3$, $\beta = 2$, and

$$\lambda_1 = \frac{1}{a_\beta} = \frac{1 + \widetilde{\chi}(0)}{\chi_\beta}, \qquad \lambda_2 = -\frac{v_\beta^2}{a_\beta} = \frac{-1}{\varepsilon_0 \mu \chi_\beta}.$$

An exact solution of (16) can be written in terms of Wright functions using Theorem 5.5 in [14]. We note that Wright functions can be represented as derivatives of the Mittag–Leffler function $E_{\alpha,\beta}[z]$ (see [14]). Solutions of (16) describe the fractional power-law damping of the magnetic field in dielectric media. An important property of the evolution described by fractional differential equations is that the solutions have fractional power-law tails.

4. Conclusion

We have proved that the electromagnetic fields and waves in a wide class of dielectric media must be described by fractional differential equations with time derivatives of the orders $2-\alpha$ and $2+\beta$, where $0<\alpha<1$ and $0<\beta<1$. The parameters $\alpha = 1-n$ and $\beta = m$ are defined by the exponents n and m in the experimentally measured frequency dependences of the dielectric susceptibility, called the universal response laws. An important property of the dynamics described by fractional differential equations for electromagnetic fields is that the solutions have fractional power-law tails. The suggested fractional differential equations for the universal electromagnetic waves in dielectrics are common (universal) to a wide class of media regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species (dipoles, electrons, or ions).

We note that the differential equations with derivatives of noninteger order proposed for describing the electromagnetic field in dielectric media can be solved numerically. For example, the Grunwald–Letnikov discretization scheme [6] is used to numerically model the electromagnetic field in dielectrics described by fractional differential equations. For small fractionality of α (or β), an ε-expansion [15] in the small parameter $\varepsilon = \alpha$ (or $\varepsilon = 1-\beta$) can be used. We note that a possible physical interpretation of fractional integrals and derivatives can be connected with memory effects or fractal properties of media (see, e.g., [16], [17]).

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