QUANTIZATION OF NON-HAMILTONIAN SYSTEMS

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In this talk a generalization of quantization which maps a dynamical operator in a function space to a dynamical superoperator in an operator space is suggested. Quantization of dynamical operator, which cannot be represented as Poisson bracket with some function, is considered. Quantization of classical systems which evolution is defined by Hamilton function is equivalent to canonical quantization. Generalized quantization of non-Hamiltonian dynamical operators is not defined by canonical quantization. Moreover the canonical quantization is a specific case of suggested quantization if dynamical operator is a operator of multiplication on a function. This approach allows to define consistent quantization procedure for non-Hamiltonian and dissipative systems. Examples of the harmonic oscillator with friction and a system which evolves by Fokker-Planck-type equation are considered.

Introduction

The quantization of non-Hamiltonian classical systems is of strong theoretical interest. As a rule, any microscopic system is always embedded in some (macroscopic) environment and therefore it is never really isolated. Frequently, the relevant environment is in principle unobservable or it is unknown [1]-[4]. This would render theory of non-Hamiltonian systems a fundamental generalization of quantum mechanics rather than an artifact of interacting with an environment [5].

We can divide the most frequent methods of quantization of non-Hamiltonian systems into two groups. The first method uses a procedure of doubling of phase-space dimension [6]-[7]. To apply the usual canonical quantization scheme to non- Hamiltonian systems, one can double the numbers of degrees of freedom, so as to deal with an effective isolated system. The new degrees of freedom may be assumed to represent by collective degrees of freedom the bath, with absorb the energy dissipated by the dissipative system [7].

The second method consists in using an explicitly time-dependent Hamiltonian [8]-[15]. It was shown that it may be possible to put the equation of motion for some non-Hamiltonian systems into time-dependent Hamiltonian form and then quantize them in the usual way [8]-[15]. However, the corresponding canonical commutation relations violate the uncertainty principle [13]. The reason for this violation would appear from the explicit dependence of Hamiltonian and momentum on the time.

To construct the canonical quantization of non-Hamiltonian systems consistently, it is possible to exceed the limits of Lie algebras and groups. The condition of self-consistency for the canonical quantization of non-Hamiltonian systems requires the application of commutant-Lie (Valya) algebra [16, 17]. Unfortunately, this algebra and its representation have not been thoroughly studied.

Note [18, 15] that Feynman wanted to develop a procedure to quantize classical equation of motion without resort to a Hamiltonian. It is interesting to quantize a classical system without direct reference to a Hamiltonian. A general classical system

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is most easily defined in terms of its equations of motion. It is difficult to determine whether a Hamiltonian exists, whether it is unique if it does exist [19, ?, 20]. Therefore, quantization that bypasses direct reference to a Poisson bracket with some Hamiltonian have practical advantages.

Canonical quantization defines a map of real functions into self-adjoint operators [21]. A classical observable is described by some real function $A(q,p)$ from a function space $\mathcal{M}$. Quantization of this function leads to self-adjoint operator $\hat{A}(\hat{q},\hat{p})$ from some operator space $\hat{\mathcal{M}}$. Quantization conserves mathematical structures (such as Lie algebra, Jordan algebra, C^*-algebra) defined on the set of observables. Note that canonical quantization of the Poisson bracket $\{\ .\ ,\ .\ \}$ leads to commutator $(-i/\hbar)[\ .\ ,\ .\ ]$. Classical state can be described by non-negative-normed function $\rho(q,p)$ called density distribution function. Quantization of a function $\rho(q,p)$ leads to non-negative self-adjoint operator $\hat{\rho}$ of trace class called matrix density operator. This description allows to consider a state as a special observable.

Time evolution of an observable $A_t(q,p)$ and a state $\rho_t(q,p)$ in classical mechanics are described by differential equations on a function space $\mathcal{M}$:

$$\frac{d}{dt}A_t(q,p) = \mathcal{L}A_t(q,p)\ , \quad \frac{d}{dt}\rho_t(q,p) = \Lambda\rho_t(q,p)\ .$$

The operators $\mathcal{L}$ and Λ, which act on the elements of function space $\mathcal{M}$, define dynamics. These operators are infinitesimal generators of dynamical semigroups and are called dynamical operators. The first equation describes evolution of an observable in the Hamilton picture, and the second equation describes evolution of a state in the Liouville picture. Dynamics of an observable and a state in quantum mechanics are described by differential equations on a operator space $\hat{\mathcal{M}}$:

$$\frac{d}{dt}\hat{A}_t(\hat{q},\hat{p}) = \hat{\mathcal{L}}\hat{A}_t(\hat{q},\hat{p})\ , \quad \frac{d}{dt}\hat{\rho}_t = \hat{\Lambda}\hat{\rho}_t\ .$$

Here $\hat{\mathcal{L}}$ and $\hat{\Lambda}$ are superoperators (operators act on operators). These superoperators are infinitesimal generators of quantum dynamical semigroups [22, 23, 24]. The first equation describes dynamics in the Heisenberg picture, and the second - in the Schroedinger picture. It is easy to see that quantization of the dynamical operators $\mathcal{L}$ and Λ leads to dynamical superoperators $\hat{\mathcal{L}}$ and $\hat{\Lambda}$. Therefore, generalization of canonical quantization must map operators into superoperators.

The usual method of quantization is applied to classical systems, where the dynamical operators have the forms $\mathcal{L}A(q,p) = \{A(q,p), H(q,p)\}$ and $\Lambda\rho(q,p) = -\{\rho(q,p), H(q,p)\}$. Here the function $H(q,p)$ is an observable which characterizes dynamics and is called the Hamilton function. Quantization of a dynamical operator which can be represented as Poisson bracket with an function is defined by the usual canonical quantization. Canonical quantization of real functions $A(q,p)$ and $H(q,p)$ leads to self-adjoint operators $\hat{A}(\hat{q},\hat{p})$ and $\hat{H}(\hat{q},\hat{p})$. Quantization of the Poisson bracket $\{A(q,p), H(q,p)\}$ leads to commutator $(i/\hbar)[\hat{H}(\hat{q},\hat{p}), \hat{A}(\hat{q},\hat{p})]$. Therefore quantization of these dynamical operators is uniquly defined by the usual canonical quantization.

Quantization of a non-Hamiltonian classical systems meets ambiguities which follow from the problems of variational description of these systems [19]-[17]. Quantization of non-Hamiltonian systems is not defined by the usual canonical quantization. Therefore, it is necessary to consider some generalization of canonical quantization. These generalized procedure must define a map of operator into superoperator [25]. The usual canonical quantization must be a specific case of generalized quantization, for quantization of operator of multiplication on a function.

In this paper quantization of non-Hamiltonian classical systems is considered. Generalization of canonical quantization, which maps an evolution equation on a function space into an evolution equation on an operator space, is suggested. An analysis of quantization is performed for operator, which cannot be represented as the Poisson bracket with some Hamilton function.

1 Canonical Quantization

Let us consider main points of the usual method of canonical quantization [21, 26, 27]. Let q^k are canonical coordinates and p^k are canonical momentums, where $k = 1, ..., n$. The basis of the space $\mathcal{M}$ of functions $A(q,p)$ is defined by functions

$$W(a,b,q,p) = e^{\frac{i}{\hbar}(aq+bp)} \ , \quad aq = \sum_{k=1}^{n} a_k q^k \ . \tag{1}$$

Quantization transforms coordinates q^k and momentums p^k to operators $\hat{q}^k$ and $\hat{p}^k$. Weyl quantization of the basis functions (1) leads to the Weyl operators

$$\hat{W}(a,b,\hat{q},\hat{p}) = e^{\frac{i}{\hbar}(a\hat{q}+b\hat{p})} \ , \quad a\hat{q} = \sum_{k=1}^{n} a_k \hat{q}^k \ . \tag{2}$$

The Weyl operators form a basis of the operator space $\hat{\mathcal{M}}$. Classical observable, characterized by the function $A(q,p)$, can be represented in the form

$$A(q,p) = \frac{1}{(2\pi\hbar)^n} \int A(a,b) W(a,b,q,p) d^n a d^n b \ , \tag{3}$$

where $A(a,b)$ is the Fourier image of the function $A(q,p)$. Quantum observable $\hat{A}(\hat{q},\hat{p})$ which corresponds to $A(q,p)$ is defined by formula

$$\hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^n} \int A(a,b) \hat{W}(a,b,\hat{q},\hat{p}) d^n a d^n b \ . \tag{4}$$

This formula can be considered as an operator expansion for $\hat{A}(\hat{q},\hat{p})$ in the operator basis (2). The direct and inverse Fourier transformations allow to write the formula (4) for the operator $\hat{A}(\hat{q},\hat{p})$ as

$$\hat{A}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^{2n}} \int A(q,p) \hat{W}(a,b,\hat{q}-qI,\hat{p}-pI) d^n a d^n b d^n q d^n p \ . \tag{5}$$

The function $A(q,p)$ is called the Weyl symbol of the operator $\hat{A}(\hat{q},\hat{p})$. Canonical quantization defined by (5) is the Weyl quantization.

Quantization must preserve some algebraic structures defined on the set $\mathcal{M}$. Lie algebra, Jordan algebra and C^*-algebra are usually considered on the spaces $\mathcal{M}$ and $\hat{\mathcal{M}}$.

Lie algebra $L(\mathcal{M})$ on the set $\mathcal{M}$ is defined by Poisson bracket

$$\{A(q,p), B(q,p)\} = \sum_{k=1}^{n} \Big(\frac{\partial A(q,p)}{\partial q^k} \frac{\partial B(q,p)}{\partial p^k} - \frac{\partial A(q,p)}{\partial p^k} \frac{\partial B(q,p)}{\partial q^k} \Big) \ . \tag{6}$$

Quantization of the Poisson bracket leads to self-adjoint commutator

$$\frac{1}{i\hbar}[\hat{A}(\hat{q},\hat{p}), \hat{B}(\hat{q},\hat{p})] = \frac{1}{i\hbar}\Big(\hat{A}(\hat{q},\hat{p})\hat{B}(\hat{q},\hat{p}) - \hat{B}(\hat{q},\hat{p})\hat{A}(\hat{q},\hat{p})\Big) \ . \tag{7}$$

The commutator defines Lie algebra $\hat{L}(\hat{\mathcal{M}})$ on the set $\hat{\mathcal{M}}$. Leibnitz rule is satisfied for the Poisson brackets. As a result, the Poisson brackets are defined by basis Poisson brackets for canonical coordinates and momentums

$$\{q^k, q^m\} = 0, \quad \{p^k, p^m\} = 0, \quad \{q^k, p^m\} = \delta_{km} \ .$$

Quantization of these relations leads to the canonical commutation relations

$$[\hat{q}^k, \hat{q}^m] = 0, \quad [\hat{p}^k, \hat{p}^m] = 0, \quad [\hat{q}^k, \hat{p}^m] = i\hbar\delta_{km} I \ . \qquad (8)$$

These relations define $(2n+1)$-parametric Lie algebra $\hat{L}(\hat{\mathcal{M}})$, called Heisenberg algebra.

Jordan algebra $J(\mathcal{M})$ for the set $\mathcal{M}$ is defined by the multiplication $A \circ B$ which coincides with the usual associative multiplication of functions. Quantization of the Jordan algebra $J(\mathcal{M})$ leads to the operator Jordan algebra $\hat{J}(\hat{\mathcal{M}})$ with multiplication

$$[\hat{A}, \hat{B}]_+ = \hat{A} \circ \hat{B} = \frac{1}{4}[(\hat{A}+\hat{B})^2 - (\hat{A}-\hat{B})^2] \ .$$

Jordan algebra for classical observables is associative algebra, that is all associators are equal to zero

$$(A \circ B) \circ C - A \circ (B \circ C) = 0 \ .$$

In general case Jordan algebra associator for quantum observables is not equal to zero

$$(\hat{A} \circ \hat{B}) \circ \hat{C} - \hat{A} \circ (\hat{B} \circ \hat{C}) = \frac{1}{4}[\hat{B}, [\hat{C}, \hat{A}]] \ .$$

This nonassociativity of the operator Jordan algebra $\hat{J}(\hat{\mathcal{M}})$ leads to an ambiguity of canonical quantization. The arbitrariness is connected with ordering of noncommutative opetators.

The Weyl operator (2) in the formula (6) leads to Weyl quantization. Another basis operator leads to different quantization scheme.

C^*-algebra can be defined on the set of quantum observables described by the bounded linear operators. In general the operator which is a result of associative multiplication of the self-adjoint operators is not self-adjoint operator. Therefore quantization of multiplication of classical observables does not leads to multiplication of the correspondent quantum observables. Universal enveloping algebra $\hat{U}(\hat{L})$ for the Lie algebra $\hat{L}(\hat{\mathcal{M}})$ which is generated by commutation relations (8) is considered as associative algebra [26, 27].

Let us consider a classical dynamical system defined by Hamilton function $H(q,p)$. Usually the quantization procedure is applied to classical systems with dynamical o perator

$$\mathcal{L} = -\{H(q,p), \ . \ \} = -\sum_{k=1}^{n}\Big(\frac{\partial H(q,p)}{\partial q^k}\frac{\partial}{\partial p^k} - \frac{\partial H(q,p)}{\partial p^k}\frac{\partial}{\partial q^k}\Big) \ . \qquad (9)$$

Here $H(q,p)$ is an observable which defines dynamics of a classical system. The observable $H(q,p)$ is called the Hamilton function. The time evolution of a classical observable is described by

$$\frac{d}{dt}A(q,p) = \{A(q,p), H(q,p)\} \ . \qquad (10)$$

If the dynamical operator has the form (9), then system is Hamiltonian system. Canonical quantization of the functions $A(q,p)$ and $H(q,p)$ leads to operators $\hat{A}(\hat{q},\hat{p})$ and $\hat{H}(\hat{q},\hat{p})$.

Canonical quantization of Poisson bracket $\{A(q,p), H(q,p)\}$ leads to the commutator $(i/\hbar)[\hat{H}(\hat{q},\hat{p}), \hat{A}(\hat{q},\hat{p})]$. Quantization of the equation (10) leads to the Heisenberg equation

$$\frac{d}{dt}\hat{A}_t(\hat{q},\hat{p}) = \frac{i}{\hbar}[\hat{H}(\hat{q},\hat{p}), \hat{A}(\hat{q},\hat{p})] \ .$$

Therefore quantization of dynamical operator (9) leads to superoperator

$$\hat{\mathcal{L}} = \frac{i}{\hbar}[\hat{H}(\hat{q},\hat{p}), \ . \ ] = \frac{i}{\hbar}(\hat{H}^l(\hat{q},\hat{p}) - \hat{H}^r(\hat{q},\hat{p})) \tag{11}$$

Here left and right superoperators $\hat{H}^l(\hat{q},\hat{p})$ and $\hat{H}^r(\hat{q},\hat{p})$ correspond to Hamilton operator $\hat{H}(\hat{q},\hat{p})$. These superoperators are defined by formulas

$$\hat{H}^l\hat{A} = \hat{H}\hat{A} \ , \quad \hat{H}^r\hat{A} = \hat{A}\hat{H} \ .$$

Quantization of dynamical operator, which can be represented as Poisson bracket with a function, is defined by canonical quantization. Therefore quantization of Hamiltonian systems is completely defined by the usual method of quantization.

2 General Dynamical System

Let us consider the time evolution of classical observable $A_t(q,p)$, described by the general differential equation

$$\frac{d}{dt}A_t(q,p) = \mathcal{L}(q,p,\partial_q,\partial_p)A_t(q,p) \ .$$

Here $\mathcal{L}(q,p,\partial_q,\partial p)$ is an operator on the function space $\mathcal{M}$. This operator cannot be expressed by Poisson bracket with a function $H(q,p)$. We would like to generalize the quantization procedure from the dynamical operators (9) to general operators $\mathcal{L}(q,p,\partial_q,\partial_p)$. In order to describe quantization of non-Hamiltonian systems we must define a general operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ using some basis operators. For simplicity, we assume that operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ is a bounded operator on the space of observables.

Let us define the basis operators which generate the dynamical operator $\mathcal{L} = \mathcal{L}(q,p,\partial_q,\partial_p)$. Operators Q_1^k and Q_2^k are operators of multiplication on coordinates q^k and p^k. Operator P_1^k and P_2^k are self-adjoint differential operator with respect to q^k and p^k, that is $P_1^k = -i\partial/\partial q^k$ and $P_2^k = -i\partial/\partial p^k$. These basis operators obey the conditions:

1. $Q_1^k 1 = q^k$, $Q_2^k 1 = p^k$ and $P_1^k 1 = 0$, $P_2^k 1 = 0$.
2. $(Q_{1,2}^k)^* = Q_{1,2}^k$, $(P_{1,2}^k)^* = P_{1,2}^k$.
3. $P_{1,2}^k(A \circ B) = (P_{1,2}^k A) \circ B + A \circ (P_{1,2}^k B)$.
4. $[Q_{1,2}^k, P_{1,2}^m] = i\delta_{km}$, $[Q_{1,2}^k, P_{2,1}^m] = 0$, $[Q_{1,2}^k, Q_{1,2}^m] = 0$, $[P_{1,2}^k, P_{1,2}^m] = 0$.

Conjugation operation $*$ is defined with respect to the usual scalar product of function space. Commutation relations for the operators $P_{1,2}^k$ and $Q_{1,2}^k$ define $(4n+1)$-parametric Lie algebra. These relations are analogous to canonical commutation relations (8) for $\hat{q}^k$ and $\hat{p}^k$ with double numbers of degrees of freedom.

Operators $Q_{1,2}^k$ and $P_{1,2}^k$ allow to introduce operator basis

$$V(a_1,a_2,b_1,b_2,Q_1,Q_2,P_1,P_2) = exp\{i(a_1Q_1 + a_2Q_2 + b_1P_1 + b_2P_2)\} \ , \tag{12}$$

for the space $\mathcal{A}(\mathcal{M})$ of dynamical operators. These basis operators are analogous to the Weyl operators (2). Note that basis functions (1) can be derived from the operators (12) by the formula

$$W(a,b,q,p) = V(a/(2\hbar), b/2\hbar), 0, 0, Q_1, Q_2, P_1, P_2)1 \ .$$

The algebra $\mathcal{A}(\mathcal{M})$ of bounded dynamical operators can be defined as C^*-algebra, generated by $Q^k_{1,2}$ and $P^k_{1,2}$. It contains all operators (12) and is closed for linear combinations of (12) in operator norm topology. A dynamical operator $\mathcal{L}$ can be defined as an operator function of basis operators $Q^k_{1,2}$ and $P^k_{1,2}$:

$$\mathcal{L}(Q_1, Q_2, P_1, P_2) = \frac{1}{(2\pi)^{2n}} \int L(a_1, a_2, b_1, b_2) e^{i(a_1 Q_1 + a_2 Q_2 + b_1 P_1 + b_2 P_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2 \, , \tag{13}$$

where $L(a_1, a_2, b_1, b_2)$ is integrable function of real variables $a_{1,2}$ and $b_{1,2}$. The function $L(a_1, a_2, b_1, b_2)$ is Fourier image of the symbol of operator $\mathcal{L} = \mathcal{L}(q, p, \partial_q, \partial_p)$. The set of bounded operators $\mathcal{L}(Q_1, Q_2, P_1, P_2)$ and their uniformly limits form the algebra $\mathcal{A}(\mathcal{M})$ of dynamical operators.

3 Quantization of Basis Operators

To define the superoperator $\hat{\mathcal{L}}$ which corresponds to operator $\mathcal{L}$ we need to describe quantization of the basis operators Q^k and P^k. Let us require that the superoperators $\hat{Q}^k$ and $\hat{P}^k$ satisfy the relations which are the quantum analogs to the relations for the operators Q^k and P^k:

1. $\hat{Q}^k_1 I = \hat{q}^k$, $\hat{Q}^k_2 I = \hat{p}^k$, and $\hat{P}^k_{1,2} I = 0$.
2. $(\hat{Q}^k_{1,2})^* = \hat{Q}^k_{1,2}$, $(\hat{P}^k_{1,2})^* = \hat{P}^k_{1,2}$.
3. $\hat{P}^k_{1,2}(\hat{A} \circ \hat{B}) = (\hat{P}^k_{1,2}\hat{A}) \circ \hat{B} + \hat{A} \circ (\hat{P}^k_{1,2}\hat{B})$.
4. $[\hat{Q}^k_{1,2}, \hat{P}^m_{1,2}] = i\delta_{km} I$, $[\hat{Q}^k_{1,2}, \hat{P}^m_{2,1}] = 0$, $[\hat{Q}^k_{1,2}, \hat{Q}^m_{1,2}] = 0$, $[\hat{P}^k_{1,2}, \hat{P}^m_{1,2}] = 0$.

Superoperator $\hat{\mathcal{L}}$ is called self-adjoint, if the relation $< \hat{\mathcal{L}}\hat{A}|\hat{B} > = < \hat{A}|\hat{\mathcal{L}}\hat{B} >$ is satisfied. The scalar product $< \hat{A}|\hat{B} >$ on the operator space $\mathcal{M}$ is defined by $< \hat{A}|\hat{B} > \equiv Sp[\hat{A}^* \hat{B}]$. An operator space with this scalar product is called Liouville space [26, 27].

To define the superoperator $\hat{P}^k_{1,2}$ we use the relations

$$P^k_1 A(q,p) = -i\frac{\partial}{\partial q^k} A(q,p) = i\{p^k, A(q,p)\} \, , \quad P^k_2 A(q,p) = -i\frac{\partial}{\partial p^k} A(q,p) = -i\{q^k, A(q,p)\} \, .$$

Canonical quantization leads to the expressions

$$\hat{P}^k_1 \hat{A}(\hat{q},\hat{p}) = \frac{1}{\hbar}[\hat{p}^k, \hat{A}(\hat{q},\hat{p})] \, , \quad \hat{P}^k_2 \hat{A}(\hat{q},\hat{p}) = -\frac{1}{\hbar}[\hat{q}^k, \hat{A}(\hat{q},\hat{p})] \, .$$

As a result, we obtain

$$\hat{P}^k_1 = \frac{1}{\hbar}[\hat{p}^k, \, . \,] = \frac{1}{\hbar}((\hat{p}^k)^l - (\hat{p}^k)^r) \, , \quad \hat{P}^k_2 = -\frac{1}{\hbar}[\hat{q}^k, \, . \,] = -\frac{1}{\hbar}((\hat{q}^k)^l - (\hat{q}^k)^r) \, , \tag{14}$$

Here we use superoperators $(\hat{q}^k)^l$, $(\hat{q}^k)^r$ and $(\hat{p}^k)^l$, $(\hat{p}^k)^r$ which satisfy the non-zero commutation relations

$$[(\hat{q}^k)^l, (\hat{p}^m)^l] = i\hbar\delta_{km} I, \qquad [(\hat{q}^k)^r, (\hat{p}^m)^r] = -i\hbar\delta_{km} I.$$

These relations follow from canonical commutation relations (8).

Let us define the self-adjoint superoperator $\hat{Q}^k_{1,2}$. Superoperators $Q^k_{1,2}$ can be defined in the form

$$\hat{Q}^k_1 = [\hat{q}^k, \, . \,]_+ = \frac{1}{2}((\hat{q}^k)^l + (\hat{q}^k)^r) \, , \quad \hat{Q}^k_2 = [\hat{p}^k, \, . \,]_+ = \frac{1}{2}((\hat{p}^k)^l + (\hat{p}^k)^r) \, , \tag{15}$$

where $\hat{Q}_1^k \hat{A} = \hat{q}^k \circ \hat{A}$ and $\hat{Q}_2^k \hat{A} = \hat{p}^k \circ \hat{A}$. There exists an arbitrariness in the definition of $Q_{1,2}^k$, which is caused by arbitrariness of the canonical quantization. This arbitrarine is connected with map of an associative Jordan algebra to a nonassociative Jordan algebra.

Quantization of the operators (12) leads to the superoperators

$$\hat{V}(a_1, a_2, b_1, b_2, \hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2) = exp\{i(a_1\hat{Q} + a_2\hat{Q}_2 + b_1\hat{P}_1 + b_2\hat{P}_2)\} . \quad (16)$$

4 Quantization of Operator Function

Let us consider the dynamical operator $\mathcal{L}$ as a function of the basis operators $Q_{1,2}^k$ and $P_{1,2}^k$. Generalized quantization can defined as a map from dynamical operator space $A(\mathcal{M})$ to dynamical superoperator space $\hat{A}(\hat{\mathcal{M}})$. This quantization of the operator

$$\mathcal{L}(Q_1, Q_2, P_1, P_2) = \frac{1}{(2\pi)^{2n}} \int L(a_1, a_2, b_1, b_2) e^{i(a_1 Q_1 + a_2 Q_2 + b_1 P_1 + b_2 P_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2 ,$$

leads to the corresponding superoperator

$$\hat{\mathcal{L}}(\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2) = \frac{1}{(2\pi)^{2n}} \int L(a_1, a_2, b_1, b_2) e^{i(a_1 \hat{Q}_1 + a_2 \hat{Q}_2 + b_1 \hat{P}_1 + b_2 \hat{P}_2)} d^n a_1 d^n a_2 d^n b_1 d^n b_2 . \quad (17)$$

If the function $L(a_1, a_2, b_1, b_2)$ is connected with Fourier image $A(a_1, a_2)$ of the function $A(q, p)$ by the relation

$$L(a_1/(2\hbar), a_2/(2\hbar), b_1, b_2) = (2\pi)^n \delta(b_1)\delta(b_2) A(a_1, a_2) ,$$

then the formula (17) defines the canonical quantization of the function $A(q, p) = \mathcal{L}(Q_1, Q_2, P_1, P_2)1$ by the relation

$$\hat{A}(\hat{q}, \hat{p}) = \hat{\mathcal{L}}(\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2) I .$$

Here we use $\hat{Q}_1^k I = \hat{q}^k$ and $\hat{Q}_2^k I = \hat{p}^k$. Therefore the usual canonical quantization is a spesific case of suggested quantization procedure.

Superoperators $\hat{Q}_{1,2}^k$ and $\hat{P}_{1,2}^k$ can be represented by $(\hat{q}^k)^l$, $(\hat{q}^k)^r$ and $(\hat{p}^k)^l$, $(\hat{p}^k)^r$. The formula (17) is written in the form

$$\hat{\mathcal{L}}(\hat{q}^l, \hat{q}^r, \hat{p}^l, \hat{p}^r) = \frac{1}{(2\pi)^{2n}} \int \tilde{L}(a_1, a_2, b_1, b_2) W^l(a_1, a_2, \hat{q}, \hat{p}) W^r(b_1, b_2, \hat{q}, \hat{p}) d^n a_1 d^n a_2 d^n b_1 d^n b_2 .$$

Here $W^l(a, b, \hat{q}, \hat{p})$ and $W^l(a, b, \hat{q}, \hat{p})$ are left and right superoperators corresponding to the Weyl operator (2). These superoperators can be defined by

$$W^l(a, b, \hat{q}, \hat{p}) = W(a, b, \hat{q}^l, \hat{p}^l) , \quad W^r(a, b, \hat{q}, \hat{p}) = W(a, b, \hat{q}^r, \hat{p}^r) .$$

We can derive [25] a relation which represents the superoperator $\hat{\mathcal{L}}$ by operator $\mathcal{L}$. We would like to write the analog of the relation (5) between an operator $\hat{A}$ and a function A. To simplify formulas, we introduce new notations. Let X^s, where $s = 1, ..., 4n$, denote the operators $Q_{1,2}^k$ and $P_{1,2}^k$, where $k = 1, ..., n$, that is

$$X^{2k-1} = q^k , \quad X^{2k} = p^k , \quad X^{2k-1+2n} = -i\frac{\partial}{\partial q^k} , \quad X^{2k+2n} = -i\frac{\partial}{\partial p^k} .$$

Let us denote the parameters $a^k_{1,2}$ and $b^k_{1,2}$, where $k = 1, ..., n$, by z^s, where $s = 1, ..., 4n$. Then the formula (13) can be rewritten by

$$\mathcal{L} = \frac{1}{(2\pi)^{2n}} \int L(z) e^{izX} d^{4n}z \ .$$

The formula (17) for the superoperator $\hat{\mathcal{L}}$ is written in the form

$$\hat{\mathcal{L}} = \frac{1}{(2\pi)^{2n}} \int L(z) e^{iz\hat{X}} d^{4n}z \ .$$

The result relation which represents the superoperator $\hat{\mathcal{L}}$ by operator $\mathcal{L}$ can be written in the form

$$\hat{\mathcal{L}} = \frac{1}{(2\pi)^{4n}} \int e^{-i\alpha(z+z')} e^{iz\hat{X}} Sp[\mathcal{L} e^{iz'\hat{X}}] d^{4n}z d^{4n}\alpha d^{4n}z' \ . \tag{18}$$

5 Harmonic oscillator with friction

Let us consider n-dimensional linear oscillator with friction $F^k_{fric} = -(\gamma/m)p^k$. The time evolution equation for this oscillator has the form

$$\frac{d}{dt} q^k = \frac{1}{m} p^k \ , \quad \frac{d}{dt} p^k = -(m\omega^2 q^k + \frac{\gamma}{m} p^k) \ . \tag{19}$$

The dynamical equation for the observable $A_t(q,p)$ is written

$$\frac{d}{dt} A_t(q,p) = \mathcal{L}(q,p,\partial_q,\partial_p) A_t(q,p) \ .$$

Differentiation of the function $A_t(q,p)$ and the formulas (19) give

$$\frac{dA_t(q,p)}{dt} = \frac{\partial A_t(q,p)}{\partial q^k} \frac{dq^k}{dt} + \frac{\partial A_t(q,p)}{\partial p^k} \frac{dp^k}{dt} = \frac{1}{m} p^k \frac{\partial A_t(q,p)}{\partial q^k} - (m\omega^2 q^k + \frac{\gamma}{m} p^k) \frac{\partial A_t(q,p)}{\partial p^k} \ .$$

The dynamical operator $\mathcal{L}(q,p,\partial_q,\partial_p)$ is

$$\mathcal{L}(q,p,\partial_q,\partial_p) = \frac{1}{m} p^k \frac{\partial}{\partial q^k} - (m\omega^2 q^k + \frac{\gamma}{m} p^k) \frac{\partial}{\partial p^k} \ .$$

QP-quantization of this operator leads to superoperator

$$\hat{\mathcal{L}} = \frac{i}{2m\hbar} [(\hat{p}^2 + m^2\omega^2\hat{q}^2)^l - (\hat{p}^2 + m^2\omega^2\hat{q}^2)^r] + \frac{i\gamma}{2m\hbar} [(\hat{p}\hat{q})^l - (\hat{p}\hat{q})^r + \hat{q}^l\hat{p}^r - \hat{p}^l\hat{q}^r] \ .$$

This superoperator can be written as $\hat{\mathcal{L}} = \hat{\mathcal{L}}_{harm} + \hat{\mathcal{L}}_{fric}$. Hence

$$\hat{\mathcal{L}}_{harm} \hat{A} = \frac{i}{\hbar} [\hat{H}, \hat{A}] \ , \quad \hat{H} = \frac{1}{2m} (\hat{p}^2 + m^2\omega^2\hat{q}^2) \ , \quad \hat{\mathcal{L}}_{fric} \hat{A} = \frac{i\gamma}{m\hbar} [\hat{p}^k, [\hat{q}^k, \hat{A}]]_+ \ .$$

QP-quantization of the dynamical operator $\mathcal{L}_{fric}$ leads to superoperator $\hat{\mathcal{L}}_{fric}$:

$$\mathcal{L}_{fric} = -\frac{\gamma}{m} p^k \frac{\partial}{\partial p^k} \quad \rightarrow \quad \hat{\mathcal{L}}_{fric} = \frac{i\gamma}{m\hbar} [\hat{p}^k, [\hat{q}^k, \ . \ ]]_+ \ .$$

6 Fokker-Planck-Type System

Let us consider Liouville operator Λ, which acts on the normed distribution density function $\rho(q,p,t)$ and has the form of second order differential operator

$$\Lambda = d_{qq}\frac{\partial^2}{\partial q^2} + 2d_{qp}\frac{\partial^2}{\partial q\partial p} + d_{pp}\frac{\partial^2}{\partial p^2} + +c_{qq}q\frac{\partial}{\partial q} + c_{qp}q\frac{\partial}{\partial p} + c_{pq}p\frac{\partial}{\partial q} + c_{pp}p\frac{\partial}{\partial p} + h \ . \quad (20)$$

Liouville equation

$$\frac{d\rho(q,p,t)}{dt} = \Lambda\rho(q,p,t)$$

is Fokker-Planck-type equation. QP-quantization of the Liouville operator (20) leads to completely dissipative superoperator $\hat{\Lambda}$, which acts on the matrix density operator

$$\hat{\Lambda} = -\frac{i}{\hbar}(\hat{H}^l - \hat{H}^r) + \frac{1}{2\hbar}\sum_{j=1,2}\left((\hat{V}_j^l - \hat{V}_j^r)\hat{V}_j^{*r} - (\hat{V}_j^{*l} - \hat{V}_j^{*r})\hat{V}_j^{*l}\right) ,$$

As the result we have the Markovian master equation [24, 26, 28]:

$$\frac{d\hat{\rho}_t}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}_t] + \frac{1}{2\hbar}\sum_{j=1,2}([\hat{V}_j\hat{\rho}_t,\hat{V}_j^*] + [\hat{V}_j,\hat{\rho}_t\hat{V}_j^*]) \ . \quad (21)$$

Here $\hat{H}$ is Hamilton operator, which has the form

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \ , \quad \hat{H}_1 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega}{2}\hat{q}^2 \ , \quad \hat{H}_2 = \frac{\mu}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) \ ,$$

where

$$m = -\frac{1}{c_{pq}} \ , \quad \omega^2 = -c_{qp}c_{pq} \ , \quad \lambda = c_{pp} + c_{qq} \ , \quad \mu = c_{pp} - c_{qq} \ .$$

Operators V_j in (21) can be written in the form $V_k = a_j\hat{p} + b_j\hat{q}$, where $j = 1,2$, and complex numbers a_j, b_j satisfy the relations

$$d_{qq} = \frac{\hbar}{2}\sum_{j=1,2}|a_j|^2 \ , \quad d_{pp} = \frac{\hbar}{2}\sum_{j=1,2}|b_j|^2 \ ,$$

$$d_{qp} = -\frac{\hbar}{2}Re(\sum_{j=1,2}a_j^* b_j) \ , \quad \lambda = -Im(\sum_{j=1,2}a_j^* b_j) \ .$$

If $h = 2c_{pp} + 2c_{qq}$, then quantum Markovian equation (21) becomes [28]:

$$\frac{d\hat{\rho}_t}{dt} = -\frac{i}{\hbar}[\hat{H}_1,\hat{\rho}_t] + \frac{i(\lambda-\mu)}{\hbar}[\hat{p},\hat{q}\circ\hat{\rho}_t] - \frac{i(\lambda+\mu)}{\hbar}[\hat{q},\hat{p}\circ\hat{\rho}_t] -$$

$$-\frac{d_{pp}}{\hbar^2}[\hat{q},[\hat{q},\hat{\rho}_t]] - \frac{d_{qq}}{\hbar^2}[\hat{p},[\hat{p},\hat{\rho}_t]] + \frac{2d_{pq}}{\hbar^2}[\hat{p},[\hat{q},\hat{\rho}_t]] \ .$$

Here d_{pp}, d_{qq}, d_{pq} are quantum diffusion coefficients and λ is a friction constant.

Conclusions

A generalization of canonical quantization procedure allows to derive dynamical superoperator from dynamical operator. Quantization of a dynamical operator which can be represented by Poisson bracket with the Hamilton function, is defined by the usual canonical quantization. Quantization of a general dynamical operator for non-Hamiltonian system cannot be described by usual canonical quantization procedure. We suggest the quantization scheme which allows to derive quantum analog for the classical non-Hamiltonian systems. The relations (17) and (18) map the operator $\mathcal{L}(q, p, \partial_q, \partial_p)$ which acts on the functions $A(q,p)$ to the superoperator $\hat{\mathcal{L}}$, which acts on the elements of operator space. If the operator $\mathcal{L}$ is an operator of multiplication by the function $A(q,p) = \mathcal{L}1$, then formula (18) defines the canonical quantization of the function $A(q,p)$ by the relation $\hat{A} = \hat{\mathcal{L}}I$. Therefore the usual canonical quantization procedure is a specific case of suggested quantization. The suggested approach allows to derive quantum analogs of chaotic dissipative sytems with strange attractors.

Canonical quantization of non-Hamiltonian systems is ambiguous which arises from problems in variational description of these systems [19]-[17]. The suggested quantization has the same arbitrariness as in the canonical quantization procedure. This arbitrariness is connected with ordering of noncommutative operators.

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КВАНТОВАНИЕ НЕГАМИЛЬТОНОВЫХ СИСТЕМ

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В данной работе предложено обобщение схемы квантования, отображающей динамический оператор в функциональном пространстве в динамический супероператор в некотором пространстве операторов. Рассмотрено квантование динамического оператора, который не может быть представлен в виде скобки Пуассона с некоторой функцией. Квантование классических систем, эволюция которых определена функцией Гамильтона, эквивалентно каноническому квантованию. Обобщенное квантование негамильтоновых динамических оераторов не является каноническим. Однако каноническое квантование является частным случаем введенной схемы квантования, если динамический оператор есть оператор умножения на функцию.

Предложенный метод позволяет определить последовательную процедуру квантования для негамильтоновых и диссипативных систем. В качестве примеров рассмотрены гармонический осциллятор с трением и система, эволюция которой определяется уравнением типа Фоккера - Планка.

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