Wave Motion 63 (2016) 18–22

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Acoustic waves in fractal media: Non-integer dimensional spaces approach

Vasily E. Tarasov

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

HIGHLIGHTS

- Acoustic waves in fractal media are proposed.
- General form of dimensions of region and boundary of region are suggested.
- Solution of acoustic wave equation for isotropic fractal media is obtained.
- Supersonic mode to locate fractal areas in materials is suggested.

ARTICLE INFO

Article history:
Received 19 August 2015
Received in revised form 14 January 2016
Accepted 19 January 2016
Available online 29 January 2016

Keywords:
Acoustic wave
Fractal media
Non-integer dimensional space
Supersonic mode

ABSTRACT

Acoustic waves in fractal media are considered in the framework of continuum models with non-integer dimensional spaces. Using recently suggested vector calculus for non-integer dimensional space, we consider waves in isotropic fractal media. The wave equation for non-integer dimensional space is similar to the equation of waves in non-fractal medium with power-law heterogeneity. We discuss some properties of speed of acoustic waves in fractal materials.

1. Introduction

Fractals can be considered as measurable metric sets with non-integer dimensions [1,2]. The fractal medium can be defined as a medium with a physical non-integer dimension (see [3] and references therein). The continuous model approach for description of fractal distributions of particles, charges, currents, media and fields has been proposed in [4–8] and then it has been developed in [9–12,3] and other works. This approach is based on the notion of power-law density of states [3]. We can consider fractal media as continuous media in non-integer dimensional space (NIDS). The non-integer dimension does not reflect all properties of the fractal media, but it is a main characteristic of fractal media. For this reason, continuous models with NIDS can allow us to get some important conclusions about the behavior of the fractal media.

Theory of integration in NIDS has been suggested in [13–15]. Stillinger introduces [13] a mathematical basis of integration on spaces with non-integer dimensions. A generalization of the Laplace operator for NIDS has been suggested in [13] also. Then Stillinger's approach [13] has been extended by Palmer and Stavrinou [15] to multiple variables, where a scalar Laplace operator for non-integer dimensional spaces is also suggested. These works propose differential operators of second order for scalar fields only, i.e. the scalar Laplacians in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian for NIDS are not considered in [13,15]. It is obvious, that consideration

E-mail address: tarasov@theory.sinp.msu.ru.

http://dx.doi.org/10.1016/j.wavemoti.2016.01.003

only the scalar Laplacian in NIDS-approach greatly restricts an application of continuous models with NIDS for fractal media and material. For example, we cannot use Stillinger's form of Laplacian for displacement vector field $\mathbf{u}(\mathbf{r}, t)$ in elasticity theories, and we cannot consider equations for the electric and magnetic fields of fractal media in the framework of NIDS models.

In work [16] the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Recently a generalization of differential vector operators of first orders (grad, div, curl), the scalar and vector Laplace operators for non-integer dimension spaces have been suggested in papers [17,18] without any approximation. This allows us to extend the application area of continuous models with NIDS. Using the suggested NIDS calculus, we can describe isotropic and anisotropic fractal media. The NIDS approach, which is based on proposed vector calculus in NIDS, has been applied in the following areas: (1) the fractal hydrodynamics to describe flow of fractal fluid in pipes [19]; (2) the fractal electrodynamics to describe fractal distribution of charges and currents [20]; (3) the theory of elasticity of fractal material [21].

In this paper, we consider acoustic waves in fractal media. Waves in fractal media have been described by continuous models in [8–12,3]. In these works the vector calculus for non-integer dimensional is not used. In this paper, we use the non-integer dimensional vector calculus, which is proposed in paper [18], to describe acoustic wave in isotropic fractal media. We prove that the wave equations for non-integer dimensional spaces are similar to the equations of waves in usual (non-fractal) media with power-law heterogeneity.

2. Vector calculus for non-integer dimensional spaces

Let us give some introduction to non-integer dimensional differentiation of integer orders (for details, see [18,17]).

In continuous models of fractal media, we will use the physically dimensionless spatial variables $x/L_0 \to x$, $y/L_0 \to x$, $z/L_0 \to x$, $\mathbf{r}/L_0 \to \mathbf{r}$, where L_0 is a characteristic size of the model. It allows us to have dimensionless differentiation in D-dimensional space such that the physical quantities of fractal media have correct physical dimensions.

For simplification, we will consider a spherically symmetric case, where scalar field Φ and vector field $\mathbf{u}$ of fractal media are independent of angles $\Phi(\mathbf{r}, t) = \Phi(r, t)$, $\mathbf{u}(\mathbf{r}, t) = u_r(r, t)\, \mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{r}|$ and $u_r = u_r(r)$ is the radial component of $\mathbf{u}$. Therefore we will use the rotationally covariant functions only. This simplification is similar to simplification, which is used in [14] for integration over NIDS.

In the general case, the dimension D of region V_D of fractal media and the dimension d of boundary $S_d = \partial V_D$ of this region are not related by the equation $d = D - 1$, i.e., $\dim(\partial V_D) \neq \dim(V_D) - 1$, where $\dim(V_D) = D$ and $\dim(\partial V_D) = d$. Let us introduce new parameter $\alpha_r = D - d$, which is a dimension of fractal medium along the radial direction.

The gradient operator for the scalar field $\Phi(\mathbf{r}) = \Phi(r)$ depends on the radial dimension α_r [18] in the form

$$\mathrm{Grad}_r^{D,d} \Phi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\, r^{\alpha_r - 1}} \frac{\partial \Phi(r)}{\partial r}\, \mathbf{e}_r. \tag{1}$$

The divergence operator of the vector field $\mathbf{u} = \mathbf{u}(r)$ can be represented [18] in the form

$$\mathrm{Div}_r^{D,d} \mathbf{u} = \pi^{(1-\alpha_r)/2} \frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{\alpha_r - 1}} \frac{\partial u_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\, u_r(r) \right). \tag{2}$$

These operators are (D, d)-dimensional gradient and divergence for fractal media with $d \neq D - 1$ (for details, see [18]). The curl operator of the vector field $\mathbf{u} = \mathbf{u}(r)$ is equal to zero, i.e., $\mathrm{Curl}_r^{D,d} \mathbf{u} = 0$.

Using operators (1) and (2) for the fields $\Phi = \Phi(r)$ and $\mathbf{u} = u(r)\, \mathbf{e}_r$, we get [18] the scalar and vector Laplace operators for NIDS with $d \neq D - 1$ in the form

$$^S\Delta_r^{D,d} \Phi = \mathrm{Div}_r^{D,d} \mathrm{Grad}_r^{D,d} \Phi, \qquad ^V\Delta_r^{D,d} \mathbf{u} = \mathrm{Grad}_r^{D,d} \mathrm{Div}_r^{D,d} \mathbf{u}. \tag{3}$$

Then the scalar Laplacian for $d \neq D - 1$ for the field $\Phi = \Phi(r)$ is defined by the equation

$$^S\Delta_r^{D,d} \Phi = \frac{\Gamma((d+\alpha_r)/2)\, \Gamma(\alpha_r/2)}{\pi^{\alpha_r - 1/2}\, \Gamma((d+1)/2)} \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 \Phi}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r - 1}} \frac{\partial \Phi}{\partial r} \right). \tag{4}$$

The vector Laplacian in non-integer dimensional space with $d \neq D - 1$ and the field $\mathbf{u} = u_r(r)\, \mathbf{e}_r$ is

$$^V\Delta_r^{D,d} \mathbf{u} = \frac{\Gamma((d+\alpha_r)/2)\, \Gamma(\alpha_r/2)}{\pi^{\alpha_r - 1/2}\, \Gamma((d+1)/2)} \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r - 1}} \frac{\partial u_r(r)}{\partial r} - \frac{d\alpha_r}{r^{2\alpha_r}}\, u_r(r) \right) \mathbf{e}_r. \tag{5}$$

For $D = 3$ and $d = 2$ Eqs. (1)–(5) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\Phi = \Phi(r)$ and $\mathbf{u}(r) = u_r(r)\, \mathbf{e}_r$.

The vector differential operators (1), (2), (4) and (5), which are suggested in [18] for NIDS, allow us to describe complex fractal media with the boundary dimension $d \neq D - 1$ by the NIDS approach.

The suggested operators allow us to reduce the NIDS vector differentiations to usual derivatives of integer orders with respect to $r = |\mathbf{r}|$. As a result, we can reduce partial differential equations for fields in NIDS to ordinary differential equations with respect to $r \geq 0$.

3. Acoustic wave equation for displacement potential

Let us consider an acoustic wave equation for the displacement potential Φ for a spherically symmetric case, where this potential is independent of angles. i.e. $\Phi(\mathbf{r}, t) = \Phi(r, t)$. The wave equation for displacement potential $\Phi(r, t)$ in NIDS has the form

$$ {}^{S}\Delta_r^{D,d} \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0, \tag{6} $$

where c is the speed of sound in non-fractal media, and ${}^{S}\Delta_r^{D,d}$ is the scalar Laplacian (4). Using expression (4), the wave equation (6) can written in the form

$$ \frac{\Gamma((d+\alpha_r)/2)\, \Gamma(\alpha_r/2)}{\pi^{\alpha_r - 1/2}\, \Gamma((d+1)/2)} \left(\frac{1}{r^{2\alpha_r - 2}} \frac{\partial^2 \Phi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r - 1}} \frac{\partial \Phi}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0. \tag{7} $$

For the case $\alpha_r = 1$ (i.e. $d = D - 1$), we have the acoustic wave equation (7) in the form

$$ \frac{\partial^2 \Phi(r)}{\partial r^2} + \frac{D-1}{r} \frac{\partial \Phi(r)}{\partial r} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0. \tag{8} $$

If $D = 3$ and $d = 2$, then $\alpha_r = 1$ and Eqs. (7), (8) have the well-known form of wave equations for non-fractal media.

Eq. (7) can be represented in the form

$$ \frac{\partial^2 \Phi}{\partial r^2} + \frac{d+1-\alpha_r}{r} \frac{\partial \Phi}{\partial r} - \frac{1}{c^2_{eff}(d, \alpha_r, r)} \frac{\partial^2 \Phi}{\partial t^2} = 0, \tag{9} $$

where c_{eff} is the effective speed of sound wave in fractal media

$$ c_{eff}(d, \alpha_r, r) = c \sqrt{\frac{\Gamma((d+\alpha_r)/2)\, \Gamma(\alpha_r/2)}{\pi^{\alpha_r - 1/2}\, \Gamma((d+1)/2)}}\, r^{1-\alpha_r}. \tag{10} $$

Let us consider a solution of the acoustic wave equation (9) in the form

$$ \Phi(r, t) = r^{\lambda}\, \varphi(r, t). \tag{11} $$

Substitution of (11) into (9) gives

$$ \frac{\partial^2 \varphi(r,t)}{\partial r^2} + \frac{2\lambda + d + 1 - \alpha_r}{r} \frac{\partial \varphi(r,t)}{\partial r} + \frac{\lambda + d - \alpha_r}{r^2} \varphi(r,t) - \frac{1}{c^2_{eff}(d, \alpha_r, r)} \frac{\partial^2 \varphi(r,t)}{\partial t^2} = 0. \tag{12} $$

The second and third terms of Eq. (12) vanish if

$$ d = \alpha_r + 1 \qquad \lambda = -1. \tag{13} $$

Note that we have two conditions on space and boundary dimensions in the form $D - d = \alpha_r = d - 1$ and $\lambda = -1$ in contrast to non-fractional case. Using (13), Eq. (12) takes the form

$$ \frac{\partial^2 \varphi(r,t)}{\partial r^2} - \frac{1}{c^2_{eff}(d, \alpha_r, r)} \frac{\partial^2 \varphi(r,t)}{\partial t^2} = 0. \tag{14} $$

We obtain the wave equation in this form only if the dimensions d and α_r are connected by the relation $d = \alpha_r + 1$. It is easy to see that Eq. (14) for fractal media with $d = \alpha_r + 1$ is similar to the equation of propagation of waves in non-fractal medium with power-law heterogeneity.

If the condition $d = \alpha_r + 1$ holds, then we can represent the effective speed of sound wave (10) in the form

$$ c_{eff}(d, \alpha_r, r) = c \sqrt{\frac{2\, \Gamma(\alpha_r + 1/2)}{\pi^{\alpha_r - 1/2}\, \alpha_r}}\, r^{1-\alpha_r}, \tag{15} $$

where we use $\Gamma(\alpha/2 + 1) = (\alpha/2)\, \Gamma(\alpha/2)$.

For the case $\alpha_r > 1$, i.e. $D > d + 1$, we can obtain a solution of Eq. (14). Let us rewrite Eq. (14) in the form

$$ x^{-2(\alpha_r - 1)} \frac{\partial^2 \varphi(r,t)}{\partial x^2} - \frac{\partial^2 \varphi(r,t)}{\partial t^2} = 0, \tag{16} $$

where $t \in \mathbb{R}$ and the variable x is defined by

$$ x = r \left(\frac{\pi^{\alpha_r - 1/2}\, \alpha_r}{2\, \Gamma(\alpha_r + 1/2)} \right)^{1/\alpha_r}. \tag{17} $$

Table 1
Modes of acoustic wave propagation in fractal media.

	$0 < r < r_{eff}$	$r = r_{eff}$	$r > r_{eff}$
$0 < \alpha_r < 1$	$c_{eff} > c$	$c_{eff} = c$	$c_{eff} < c$
$\alpha_r = 1$	$c_{eff} = c$	$c_{eff} = c$	$c_{eff} = c$
$\alpha_r > 1$	$c_{eff} < c$	$c_{eff} = c$	$c_{eff} > c$

Let us assume that

$$\varphi(0,t) = \varphi_0(t), \qquad \left(\frac{\partial \varphi}{\partial x}\right)_{x=0} = \psi_0(t). \tag{18}$$

Then, using equation 5.3.4.13 of [22,23] for (16), we obtain a solution of the acoustic wave equation (9) with $d = \alpha_r + 1$ in the form (11) with

$$\varphi(x,t) = \frac{\Gamma(2\,a_r)}{\Gamma^2(a_r)} \int_0^1 \varphi_0\left(t + \frac{1}{\alpha_r} x^{\alpha_r}(2\xi - 1)\right)(\xi\,(1-\xi))^{a_r - 1}\, d\xi$$
$$+ \frac{\Gamma(2 - 2a_r)}{\Gamma^2(1 - a_r)} \int_0^1 \psi_0\left(t + \frac{1}{\alpha_r} x^{\alpha_r}(2\xi - 1)\right)(\xi\,(1-\xi))^{-a_r}\, d\xi, \tag{19}$$

where

$$a_r = \frac{\alpha_r - 1}{2\,\alpha_r}. \tag{20}$$

As a simple example of fractals with $\alpha_r < 1$ and $\alpha_r > 1$, we can consider well-known fractal sets such as the Cantor set and the Koch curve [1,2]. The Cantor set is defined by repeatedly removing the middle thirds of line segments. The fractal dimension of the Cantor set is $\alpha = \log_3(2) \approx 0.631 < 1$. The Koch curve is defined by repeatedly removing the middle thirds of line segments and then replacing this interval by equilateral triangle without this segment. The fractal dimension of the Koch curve is $\alpha = \log_3(4) \approx 1.262 > 1$.

Let us discuss the effective speed of acoustic waves $c_{eff}(d, \alpha_r, r)$, which is defined by Eqs. (10) and (15) for fractal media in the framework of NIDS approach. There are three cases that correspond to the conditions (I) $c_{eff} \le c$; (II) $c_{eff} = c$; (III) $c_{eff} \ge c$. To describe these cases, we introduce the effective distance

$$r_{eff}(d, \alpha_r) = \left(\frac{2\,\Gamma(\alpha_r + 1/2)}{\pi^{\alpha_r - 1/2}\,\alpha_r}\right)^{1/(2-2\alpha_r)}. \tag{21}$$

(I) For all $\alpha_r > 1$ and $d = \alpha_r + 1$, we have $c_{eff} \le c$ if $r \ge r_{eff}(d, \alpha_r)$. For all $0 < \alpha_r < 1$ and $d = \alpha_r + 1$, we have $c_{eff} \le c$ if $r \le r_{eff}(d, \alpha_r)$.
(II) Using Eq. (15), it is easy to see that $c_{eff} = c$ if $\alpha_r = 1$, since $\Gamma(1/2) = \sqrt{\pi}$.
(III) For the cases: (a) $\alpha_r > 1$, $d = \alpha_r + 1$ and $r \le r_{eff}(d, \alpha_r)$, (b) $0 < \alpha_r < 1$, $d = \alpha_r + 1$ and $r \ge r_{eff}(d, \alpha_r)$, we have $c_{eff} \ge c$ that can be considered as an effective supersonic mode caused by non-integer dimensionality of space or fractality. The effective supersonic mode is the acoustic wave with an effective speed exceeding the sound speed for non-fractal material.

These three cases are presented by Table 1.

As a result, we see that the existence of the supersonic modes is characteristic property of fractal models of media and models with NIDS, if we use that fields exist in fractal set only. Obviously the effective supersonic mode cased by non-integer dimensionality of space does not exist if the acoustic waves can travel not only in fractal media. At the same time, acoustic waves, unlike electromagnetic waves, require the presence of a material medium in order to transport their energy from one location to another. Therefore acoustic waves exist only in fractal media, and the supersonic modes can be observed. We assume that the supersonic modes can be used to locate fractal areas of homogeneous materials: If the acoustic wave travels a path between two spatial points A and B of homogeneous materials in a time $t < |AB|/c$, then we can conclude that there is a fractal region between these points.

4. Conclusion

The vector calculus for non-integer dimensional space (NIDS) has been suggested in recent papers [18,17]. This calculus includes generalizations of differential vector operators of first orders (gradient, divergence, curl operators), the scalar and vector Laplace operators. It allows us to describe fractal media for isotropic and anisotropic cases by using continuous models with NIDS. In this paper, we use the vector calculus for NIDS, which is proposed in paper [18], to describe acoustic waves in fractal media and materials.

It should be noted that the vector calculus of [18] cannot be applied for fractal media and fields in anisotropic cases. For anisotropic fractal media, we should use NIDS calculus, which is suggested in [17]. New approach to generalize grad, div, curl operators and the corresponding scalar and vector Laplace operators for anisotropic fractal case have been proposed in paper [17] without any approximations in contrast with [16]. This NIDS calculus can be used for more rigorous description of anisotropic fractal media and fields by continuous models with NIDS.

The suggested concept of supersonic modes can be used to locate fractal areas in materials and media. The suggested approach also allows us to describe materials with higher sound insulation.

References

[1] J. Feder, Fractals, Plenum Press, New York, London, 1988.
[2] K.F. Falconer, The Geometry of Fractal Sets, Cambridge University Press, Cambridge, 1985.
[3] V.E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, New York, 2011.
[4] V.E. Tarasov, Continuous medium model for fractal media, Phys. Lett. A 336 (2–3) (2005) 167–174. arXiv:cond-mat/0506137.
[5] V.E. Tarasov, Possible experimental test of continuous medium model for fractal media, Phys. Lett. A 341 (5–6) (2005) 467–472. arXiv:physics/0602121.
[6] V.E. Tarasov, Fractional hydrodynamic equations for fractal media, Ann. Physics 318 (2) (2005) 286–307. arXiv:physics/0602096.
[7] V.E. Tarasov, Dynamics of fractal solid, Internat. J. Modern Phys. B 19 (27) (2005) 4103–4114. arXiv:0710.0787.
[8] V.E. Tarasov, Wave equation for fractal solid string, Modern Phys. Lett. B 19 (15) (2005) 721–728. arXiv:physics/0605006.
[9] J. Li, M. Ostoja-Starzewski, Fractal solids, product measures and fractional wave equations, Proc. R. Soc. A 465 (2108) (2009) 2521–2536; J. Li, M. Ostoja-Starzewski, Correction to Li and Ostoja-Starzewski 465 (2108) 2521, Proc. R. Soc. A 467 (2128) (2011) 1214 (1 page).
[10] P.N. Demmie, M. Ostoja-Starzewski, Waves in fractal media, J. Elasticity 104 (1–2) (2011) 187–204.
[11] H. Joumaa, M. Ostoja-Starzewski, On the wave propagation in isotropic fractal media, Z. Angew. Math. Phys. 62 (6) (2011) 1117–1129.
[12] M. Ostoja-Starzewski, J. Li, H. Joumaa, P.N. Demmie, From fractal media to continuum mechanics, Z. Angew. Math. Mech. 94 (5) (2014) 373–401.
[13] F.H. Stillinger, Axiomatic basis for spaces with noninteger dimensions, J. Math. Phys. 18 (6) (1977) 1224–1234.
[14] J.C. Collins, Renormalization, Cambridge University Press, Cambridge, 1984.
[15] C. Palmer, P.N. Stavrinou, Equations of motion in a non-integer-dimensional space, J. Phys. A 37 (27) (2004) 6987–7003.
[16] M. Zubair, M.J. Mughal, Q.A. Naqvi, Electromagnetic Fields and Waves in Fractional Dimensional Space, Springer, Berlin, 2012.
[17] V.E. Tarasov, Anisotropic fractal media by vector calculus in non-integer dimensional space, J. Math. Phys. 55 (8) (2014) 083510. arXiv:1503.02392.
[18] V.E. Tarasov, Vector calculus in non-integer dimensional space and its applications to fractal media, Commun. Nonlinear Sci. Numer. Simul. 20 (2) (2015) 360–374. arXiv:1503.02022.
[19] V.E. Tarasov, Flow of fractal fluid in pipes: Non-integer dimensional space approach, Chaos Solitons Fractals 67 (2014) 26–37. arXiv:1503.02842.
[20] V.E. Tarasov, Fractal electrodynamics via non-integer dimensional space approach, Phys. Lett. A 379 (36) (2015) 2055–2061.
[21] V.E. Tarasov, Elasticity of fractal material by continuum model with non-integer dimensional space, C.R. Mec. 343 (1) (2015) 57–73. arXiv:1503.03060.
[22] A.D. Polyanin, Handbook of Linear Partial Differential Equations, Chapman, Boca Raton, 2002.
[23] V.F. Zaitsev, A.D. Polyanin, Handbook on Differential Equations, International Education Program, Moscow, 1996, p. 496. ISBN: 5-7753-0017-3 (in Russian).