Pure stationary states of open quantum systems

Vasily E. Tarasov*
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia

(Received 19 September 2001; revised manuscript received 12 September 2002; published 19 November 2002)

Using Liouville space and superoperator formalism we consider pure stationary states of open and dissipative quantum systems. We discuss stationary states of open quantum systems, which coincide with stationary states of closed quantum systems. Open quantum systems with pure stationary states of linear oscillator are suggested. We consider stationary states for the Lindblad equation. We discuss bifurcations of pure stationary states for open quantum systems which are quantum analogs of classical dynamical bifurcations.

DOI: 10.1103/PhysRevE.66.056116 PACS number(s): 05.30.Ch, 03.65.−w, 03.65.Yz

I. INTRODUCTION

The open quantum systems are of strong theoretical interest. As a rule, any microscopic system is always embedded in some (macroscopic) environment and therefore it is never really closed. Frequently, the relevant environment is in principle unobservable or it is unknown [1,2]. This would render the theory of open quantum systems a fundamental generalization of quantum mechanics [3,4].

Classical open and dissipative systems can have regular or strange attractors [5,6]. Regular attractors can be considered as a set of (stationary) states for closed classical systems correspondent to open systems. Quantization of evolution equations in phase space for dissipative and open classical systems was suggested in Refs. [7,8]. This quantization procedure allows one to derive quantum analogs of open classical systems with regular attractors such as nonlinear oscillator [7,9]. In the papers [7–9] were derived quantum analogs of dissipative systems with strange attractors such as Lorenz-like system, Ressler and Newton-Leipnik systems. It is interesting to consider quantum analogs for regular and strange attractors. The regular "quantum" attractors can be considered as stationary states of open quantum systems. The existence of stationary states for open quantum systems is an interesting fact. The condition given by Davies in Ref. [10] defines the stationary state of an open quantum system. An example, where the stationary state is unique and approached by all states for long times is considered by Lindblad [11] for Brownian motion of quantum harmonic oscillator. In Refs. [12–14] Spohn derives sufficient condition for the existence of a unique stationary state for the open quantum system described by Lindblad equation [15]. The stationary solution of the Wigner function evolution equation for an open quantum system was discussed in Refs. [16,17]. Quantum effects in the steady states of the dissipative map are considered in Ref. [18].

In this paper we consider stationary pure states of some open quantum systems. These open systems look like closed quantum systems in the pure stationary states. We consider the quantum analog of dynamical bifurcations considered by Thompson and Lunn [19] for classical dynamical systems. In order to describe these systems, we consider Liouville–von Neumann equation for density matrix evolution such that this Liouville generator of the equation is a function of some Hamiltonian operator. Open quantum systems with pure stationary states of linear harmonic oscillator are suggested. We derive stationary states for quantum Markovian master equation usually called the Lindblad equation. The suggested approach allows one to use theory of bifurcations for a wide class of quantum open systems. We consider the example of bifurcation of pure stationary states for open quantum systems.

In Sec. II, we introduce pure stationary states for closed quantum systems and some mathematical background is considered. In Sec. III, we consider Liouville–von Neumann equation for an open quantum system and pure stationary states for this equation. In Sec. IV, simple examples of stationary states for open quantum systems are considered. In Sec. V, we study some proporties of the quantum system to have dynamical bifurcations and catastrophes. In the Sec. VI, we suggest an example of the quantum system with fold catastrophe. Finally, a short conclusion is given in Sec. VII. In the Appendix, the mathematical background (Liouville space, superoperators) is considered.

II. PURE STATIONARY STATE

In the general case, the time evolution of the quantum state $|\rho_t)$ can be described by the Liouville–von Neumann equation

$$\frac{d}{dt}|\rho_t)=\hat{\Lambda}|\rho_t), \tag{1}$$

where $\hat{\Lambda}$ is a Liouville superoperator on Liouville space, $|\rho)$ is a density matrix operator as an element of Liouville space. For the concept of Liouville space and superoperators see the Appendix and Refs. [20–37]. For closed systems, the Liouville superoperator has the form

$$\hat{\Lambda}=-\frac{i}{\hbar}(\hat{L}_H-\hat{R}_H) \quad \text{or} \quad \hat{\Lambda}=\hat{L}_H^-, \tag{2}$$

where $H=H(q,p)$ is a Hamilton operator. If the Liouville superoperator $\hat{\Lambda}$ cannot be represented in the form (2), then

*Email address: tarasov@theory.sinp.msu.ru

quantum system is called open, non-Hamiltonian or dissipative quantum system [36–38]. The stationary state is defined by the following condition:

$$\hat{\Lambda}|\rho_t)=0. \tag{3}$$

For closed quantum systems (2), this condition has the simple form

$$\hat{L}_H|\rho_t)=\hat{R}_H|\rho_t) \quad \text{or} \quad \hat{L}_H^-|\rho_t)=0. \tag{4}$$

In the general case, we can consider the Liouville superoperator as a superoperator function [7,8,36]:

$$\hat{\Lambda}=\Lambda(\hat{L}_X^-,\hat{L}_X^+) \quad \text{or} \quad \hat{\Lambda}=\Lambda(\hat{L}_X,\hat{R}_X),$$

where X is a set of linear operators. For example, $X=\{q,p,H\}$ or $X=\{H_1,\ldots,H_s\}$. In this paper we use the special form of the superoperator $\hat{\Lambda}$ such that

$$\hat{\Lambda}=-\frac{i}{\hbar}(\hat{L}_H-\hat{R}_H)+\sum_{k=1}^{s}\hat{F}_kN_k(\hat{L}_H,\hat{R}_H),$$

where $N_k(\hat{L}_H,\hat{R}_H)$ are some superoperator functions and $\hat{F}^k$ is an arbitrary nonzero superoperator.

It is known that a pure state $|\rho_\Psi)$ is a stationary state of a closed quantum system [Eqs. (1), (2)], if the state $|\rho_\Psi)$ is an eigenvector of the Liouville space for superoperators $\hat{L}_H$ and $\hat{R}_H$:

$$\hat{L}_H|\rho_\Psi)=|\rho_\Psi)E, \quad \hat{R}_H|\rho_\Psi)=|\rho_\Psi)E. \tag{5}$$

Equivalently, the state $|\rho_\Psi)$ is an eigenvector of superoperators L_H^+ and L_H^- such that

$$\hat{L}_H^+|\rho_\Psi)=|\rho_\Psi)E, \quad \hat{L}_H^-|\rho_\Psi)=|\rho_\Psi)\cdot 0=0.$$

The energy variable E can be defined by

$$E=(I|\hat{L}_H|\rho_\Psi)=(I|\hat{R}_H|\rho_\Psi)=(I|\hat{L}_H^+|\rho_\Psi).$$

The superoperators $\hat{L}_H$ and $\hat{R}_H$ for linear harmonic oscillator are

$$\hat{L}_H=\frac{1}{2m}\hat{L}_p^2+\frac{m\omega^2}{2}\hat{L}_q^2, \quad \hat{R}_H=\frac{1}{2m}\hat{R}_p^2+\frac{m\omega^2}{2}\hat{R}_q^2. \tag{6}$$

It is known that pure stationary states $\rho_{\Psi_n}=\rho_{\Psi_n}^2$ of linear harmonic oscillator (6) exists if the variable E is equal to

$$E_n=\tfrac{1}{2}\hbar\omega(2n+1). \tag{7}$$

III. PURE STATIONARY STATES OF OPEN SYSTEMS

Let us consider the Liouville–von Neumann equation (1) for the open quantum system defined of the form

$$\frac{d}{dt}|\rho_t)=-\frac{i}{\hbar}(\hat{L}_H-\hat{R}_H)|\rho_t)+\sum_{k=1}^{s}\hat{F}_kN_k(\hat{L}_H,\hat{R}_H)|\rho_t). \tag{8}$$

Here $\hat{F}^k$ is some superoperator and $N_k(\hat{L}_H,\hat{R}_H)$, where $k=1,\ldots,s$, are superoperator functions.

Let $|\rho_\Psi)$ be a pure stationary state of the closed quantum system defined by Hamilton operator H. If Eqs. (5) are satisfied, then the state $|\rho_\Psi)$ is a stationary state of the closed system associated with the open system (8) and is defined by

$$\frac{d}{dt}|\rho_t)=-\frac{i}{\hbar}(\hat{L}_H-\hat{R}_H)|\rho_t). \tag{9}$$

If the vector $|\rho_\Psi)$ is an eigenvector of operators $\hat{L}_H$ and $\hat{R}_H$, then the Liouville–von Neumann equation (8) for the pure state $|\rho_\Psi)$ has the form

$$\frac{d}{dt}|\rho_\Psi)=\sum_{k=1}^{s}\hat{F}_k|\rho_\Psi)N_k(E,E),$$

where the functions $N_k(E,E)$ are defined by

$$N_k(E,E)=(I|N_k(\hat{L}_H,\hat{R}_H)|\rho_\Psi).$$

If all functions $N_k(E,E)$ are equal to zero

$$N_k(E,E)=0, \tag{10}$$

then the stationary state $|\rho_\Psi)$ of the closed quantum system (9) is the stationary state of the open quantum system (8).

Note that functions $N_k(E,E)$ are eigenvalues and $|\rho_\Psi)$ is the eigenvector of superoperators $N_k(\hat{L}_H,\hat{R}_H)$, since

$$N_k(\hat{L}_H,\hat{R}_H)|\rho_\Psi)=|\rho_\Psi)N_k(E,E).$$

Therefore stationary states of the open quantum system (8) are defined by zero eigenvalues of superoperators $N_k(\hat{L}_H,\hat{R}_H)$.

IV. OPEN SYSTEMS WITH OSCILLATOR STATIONARY STATES

In this section, simple examples of open quantum systems (8) are considered.

(1) Let us consider the nonlinear oscillator with friction defined by the equation

$$\frac{d}{dt}\rho_t=-\frac{i}{\hbar}[\tilde{H},\rho_t]-\frac{i}{2\hbar}\beta[q^2,p^2\rho_t+\rho_tp^2], \tag{11}$$

where the operator $\tilde{H}$ is the Hamiltom operator of the nonlinear oscillator:

$$\tilde{H}=\frac{p^2}{2m}+\frac{m\Omega^2q^2}{2}+\frac{\gamma q^4}{2}.$$

Equation (11) can be rewritten in the form

$$\frac{d}{dt}|\rho_t)=\hat{L}_H^-|\rho_t)+2m\beta\hat{L}_{q^2}^- \times\left(\frac{1}{2m}(\hat{L}_p^+)^2+\frac{\gamma}{2m\beta}(\hat{L}_q^+)^2-\frac{\Delta}{4\beta}\hat{L}_I\right)|\rho_t), \tag{12}$$

where $\Delta=\Omega^2-\omega^2$ and the superoperator $\hat{L}_H^-$ is defined for the Hamilton operator H of the linear harmonic oscillator by Eqs. (2) and (6). Equation (12) has the form (8), with

$$N(\hat{L}_H,\hat{R}_H)=\frac{1}{2}(\hat{L}_H+\hat{R}_H)-\frac{\Delta}{2\beta}\hat{L}_I, \quad \hat{F}=2m\beta\hat{L}_{q^2}^-.$$

In this case the function $N(E,E)$ has the form

$$N(E,E)=E-\frac{\Delta}{2\beta},$$

Let $\gamma=\beta m^2\omega^2$. The open quantum system (11) has one stationary state of the linear harmonic oscillator with energy $E_n=(\hbar\omega/2)(2n+1)$, if $\Delta=2\beta\hbar\omega(2n+1)$, where n is an integer non-negative number. This stationary state is one of the stationary states of the linear harmonic oscillator with the mass m and frequency ω. In this case we can have the quantum analog of dynamical Hopf bifurcation [19,39].

(2) Let us consider the open quantum system described by the time evolution equation

$$\frac{d}{dt}|\rho_t)=\hat{L}_H^-|\rho_t)+\hat{L}_q^-\cos\left(\frac{\pi}{\varepsilon_0}\hat{L}_H^+\right)|\rho_t), \tag{13}$$

where the superoperator $\hat{L}_H^-$ is defined by formulas (2) and (6). Equation (13) has the form (8) if the superoperators $\hat{F}$ and $N(\hat{L}_H,\hat{R}_H)$ are defined by

$$\hat{F}=-\frac{i}{\hbar}(\hat{L}_q-\hat{R}_q),$$

$$N(\hat{L}_H,\hat{R}_H)=\cos\left(\frac{\pi}{2\varepsilon_0}(\hat{L}_H+\hat{R}_H)\right) =\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(\frac{i\pi}{2\varepsilon_0}\right)^{2m}(\hat{L}_H+\hat{R}_H)^{2m}. \tag{14}$$

The function $N(E,E)$ has the form

$$N(E,E)=\cos\left(\frac{\pi E}{\varepsilon_0}\right)=\sum_{m=0}^{\infty}\frac{1}{(2m)!}\left(\frac{i\pi E}{\varepsilon_0}\right)^{2m}.$$

The stationary state condition (10) has the solution

$$E=\frac{\varepsilon_0}{2}(2n+1),$$

where n is an integer number. If parameter ε_0 is equal to $\hbar\omega$, then quantum systems (13) and (14) have pure stationary states of the linear harmonic oscillator with the energy (7). As the result, stationary states of open the quantum system (13) coincide with pure stationary states of the linear harmonic oscillator. If the parameter ε_0 is equal to $\hbar\omega(2m+1)$, then quantum systems (13) and (14) have stationary states of the linear harmonic oscillator with $n(k,m)=2km+k+m$ and

$$E_{n(k,m)}=\frac{\hbar\omega}{2}(2k+1)(2m+1).$$

(3) Let us consider the superoperator function $N_k(\hat{L}_H,\hat{R}_H)$ in the form

$$N_k(\hat{L}_H,\hat{R}_H)=\frac{1}{2\hbar}\sum_{n,m}v_{kn}v_{km}^*(2\hat{L}_H^n\hat{R}_H^m-\hat{L}_H^{n+m}-\hat{R}_H^{n+m}),$$

and all superoperators $\hat{F}_k$ are equal to $\hat{L}_I$. In this case, the Liouville–von Neumann equation (8) can be represented by the Lindblad equation [40,41,36]:

$$\frac{d}{dt}|\rho_t)=-\frac{i}{\hbar}(\hat{L}_H-\hat{R}_H)|\rho_t)+\frac{1}{2\hbar}\sum_j(2\hat{L}_{V_k}\hat{R}_{V_k^\dagger}-\hat{L}_{V_k}\hat{L}_{V_k^\dagger} -\hat{R}_{V_k^\dagger}\hat{R}_{V_k})|\rho_t), \tag{15}$$

with linear operators V_k defined by

$$V_k=\sum_n v_{kn}H^n, \quad V_k^\dagger=\sum_m v_{km}^*H^m. \tag{16}$$

If $|\rho_\Psi)$ is a pure stationary state (5), then all functions $N_k(E,E)$ are equal to zero and this state $|\rho_\Psi)$ is a stationary state of the open quantum system (15).

If the Hamilton operator H is defined by

$$H=\frac{1}{2m}p^2+\frac{m\omega^2}{2}q^2+\frac{\lambda}{2}(qp+pq), \tag{17}$$

then we have some generalization of the quantum model for the Brownian motion of a harmonic oscillator considered in Ref. [11]. Note that in the model [11] operators V_k are linear $V_k=a_kp+b_kq$, but in our generalizations (15) and (16) these operators are nonlinear. For example, we can use

$$V_k=a_kp+b_kq+c_kp^2+d_kq^2+e_k(qp+pq).$$

The case $c_k=d_k=e_k=0$ is considered in Ref. [11]. Let real parameters α and β exist and

$$b_k=\alpha a_k, \quad c_k=\beta a_k, \quad d_k=m^2\omega^2\beta a_k, \quad e_k=m\lambda\beta a_k.$$

In this case, the pure stationary states of the linear oscillator (17) exist if $\omega>\lambda$ and the variable E is equal to

$$E_n=\tfrac{1}{2}\hbar\omega(2n+1)\sqrt{1-\lambda^2/\omega^2}.$$

V. DYNAMICAL BIFURCATIONS AND CATASTROPHES

Let us consider a special case of open quantum systems (8) such that the vector function

$$N_k(E,E)=(I|N_k(\hat{L}_H,\hat{R}_H)|\rho)$$

be a potential function and the Hamilton operator H can be represented in the form

$$H=\sum_{k=1}^{s} H_k .$$

In this case we have a function $V(E)$ called potential, such that the following conditions are satisfied:

$$\frac{\partial V(E)}{\partial E_k}=N_k(E,E),$$

where $E_k=(I|\hat{L}_{H_k}|\rho)=(I|\hat{R}_{H_k}|\rho)$. If potential $V(E)$ exists, then the stationary state condition (10) for the open quantum system (8) is defined by critical points of the potential $V(E)$. If the system has one variable E, then the function $N(E,E)$ is always a potential function. In general, the vector function $N_k(E,E)$ is potential, if

$$\frac{\partial N_k(E,E)}{\partial E_l}=\frac{\partial N_l(E,E)}{\partial E_k}.$$

Stationary states of the open quantum system (8) with the potential vector function $N_k(E,E)$ depend on critical points of the potential $V(E)$. It allows one to use the theory of bifurcations and catastrophes for the parametric set of functions $V(E)$. Note that a bifurcation in a vector space of variables $E=\{E_k|k=1,\ldots,s\}$ is a bifurcation in the vector space of eigenvalues of the Hamilton operator H_k.

For the polynomial superoperator function $N_k(\hat{L}_H,\hat{R}_H)$, we have

$$N_k(\hat{L}_H,\hat{R}_H)=\sum_{n=0}^{N}\sum_{m=0}^{n} a_{n,m}^{(k)}\hat{L}_H^m\hat{R}_H^{n-m} .$$

In general, m and n are multi-indices. The function $N_k(E,E)$ is a polynomial,

$$N_k(E,E)=\sum_{n=0}^{N} \alpha_n^{(k)}E^n,$$

where the coefficients $\alpha_n^{(k)}$ are defined by

$$\alpha_n^{(k)}=\sum_{m=0}^{n} a_{n,m}^{(k)} .$$

We can define the variables $x_l=E_l-a_l$ $(l=1,\ldots,s)$, such that functions $N_k(E,E)=N_k(x+a,x+a)$ do not have the terms x_l^{n-1}.

$$N_k(x+a,x+a)=\sum_{n=0}^{N} \alpha_n^{(k)}(x+a^{(k)})^n = \sum_{n=0}^{N}\sum_{m=0}^{n} \alpha_n^{(k)}\frac{n!}{m!(n-m)!}x^m(a^{(k)})^{n-m}.$$

If the coefficient of the term $x_l^{n_l-1}$ is equal to zero

$$\alpha_{n_l}^{(k)}\frac{n_l!}{(n_l-1)!}a_l^{(k)}+\alpha_{n_l-1}^{(k)}=\alpha_{n_l}^{(k)}n_l a_l^{(k)}+\alpha_{n_l-1}^{(k)}=0,$$

then we have the following coefficients:

$$a_l^{(k)}=-\frac{\alpha_{n_l-1}^{(k)}}{n_l\alpha_{n_l}^{(k)}}.$$

If we change parameters $\alpha_n^{(k)}$, then an open quantum system can have pure stationary states of the system. For example, the bifurcation with the birth of linear oscillator pure stationary state is a quantum analog of dynamical Hopf bifurcation [19,39] for a classical dynamical system.

Let a vector space of energy variables E be a one-dimensional space. If the function $N(E,E)$ is equal to

$$N(E,E)=\pm\alpha_n E^n+\sum_{j=1}^{n-1} \alpha_j E^j, \qquad n\geqslant 2,$$

then the potential $V(x)$ is defined by the following equation:

$$V(x)=\pm x^{n+1}+\sum_{j=1}^{n-1} a_j x^j, \qquad n\geqslant 2,$$

and we have catastrophe of type $A_{\pm n}$.

If we have s variables E_l, where $l=1,2,\ldots,s$, then quantum analogs of elementary catastrophes $A_{\pm n}$, $D_{\pm n}$, $E_{\pm 6}$, E_7, and E_8 can be realized for open quantum systems. Let us write the full list of potentials $V(x)$, which leads to elementary catastrophes (zero modal) defined by $V(x)=V_0(x)+Q(x)$, where

$$A_{\pm n}:\; V_0(x)=\pm x_1^{n+1}+\sum_{j=1}^{n-1} a_j x_1^j, \quad n\geqslant 2,$$

$$D_{\pm n}:\; V_0(x)=x_1^2x_2\pm x_2^{n-1}+\sum_{j=1}^{n-3} a_j x_2^j+\sum_{j=n-2}^{n-1} x_1^{j-(n-3)},$$

$$E_{\pm 6}:\; V_0(x)=(x_1^3\pm x_2^4)+\sum_{j=1}^{2} a_j x_2^j+\sum_{j=3}^{5} a_j x_1 x_2^{j-3},$$

$$E_7:\; V_0(x)=x_1^3+x_1x_2^3+\sum_{j=1}^{4} a_j x_2^j+\sum_{j=5}^{6} a_j x_1 x_2^{j-5},$$

$$E_8:\; V_0(x)=x_1^3+x_2^5+\sum_{j=1}^{3} a_j x_2^j+\sum_{j=4}^{7} a_j x_1 x_2^{j-4}.$$

Here $Q(x)$ is the nondegenerate quadratic form with variables $x_2,x_3,\ldots,x_s$ for $A_{\pm n}$ and parameters $x_3,\ldots,x_s$ for other cases.

VI. FOLD CATASTROPHE

In this section, we suggest an example of the open quantum system with catastrophe A_2 called fold.

Let us consider the Liouville–von Neumann equation (8) for a nonlinear quantum oscillator with friction, where multiplication superoperators $\hat{L}_H$ and $\hat{R}_H$ are defined by Eq. (6) and superoperators $\hat{F}$ and $N(\hat{L}_H,\hat{R}_H)$ are given by the following equations:

$$\hat{F}=-2\hat{L}_q^-\hat{L}_p^+, \tag{18}$$

$$N(\hat{L}_H,\hat{R}_H)=\alpha_0\hat{L}_I^+ + \alpha_1\hat{L}_H^+ + \alpha_2(\hat{L}_H^+)^2. \tag{19}$$

In this case, the function $N(E,E)$ is equal to

$$N(E,E)=\alpha_0+\alpha_1 E+\alpha_2 E^2.$$

A pure stationary state $|\rho_\Psi)$ of the linear harmonic oscillator is a stationary state of the open quantum system (19), if $N(E,E)=0$. Let us define the new real variable x and parameter λ by the following equation:

$$x=E+\frac{\alpha_1}{2\alpha_2},\quad \lambda=\frac{4\alpha_0\alpha_2-\alpha_1^2}{4\alpha_2^2}.$$

Then we have the stationary condition $N(E,E)=0$ in the form $x^2-\lambda=0$. If $\lambda\leqslant 0$, then the open quantum system has no stationary states. If $\lambda>0$, then we have pure stationary states for a discrete set of parameter values λ. If parameters α_1,α_2, and λ satisfy the following conditions

$$-\frac{\alpha_1}{2\alpha_2}=\hbar\omega\left(n+\frac{1}{2}+\frac{m}{2}\right),\quad \lambda=\hbar^2\omega^2\frac{m^2}{4},$$

where n and m are non-negative integer numbers, then the open quantum systems (18) and (19) have two pure stationary states of the linear harmonic oscillator. The energies of these states are equal to

$$E_n=\hbar\omega(n+\tfrac{1}{2}),\quad E_{n+m}=\hbar\omega(n+m+\tfrac{1}{2}).$$

VII. CONCLUSION

Open quantum systems can have pure stationary states. Stationary states of open quantum systems can coincide with pure stationary states of closed (Hamiltonian) systems. As an example, we suggest open quantum systems with pure stationary states of linear oscillator. Note that using Eq. (8), it is easy to get open (dissipative) quantum systems with stationary states of the hydrogen atom. For a special case of open systems, we can use usual bifurcation and catastrophe theory. It is easy to derive quantum analogs of classical dynamical bifurcations. Physical instances with bifurcation behavior can be realized in quantum optics [42] and deep inelastic collisions [43].

Open quantum systems with two stationary states can be considered as qubits. It allows one to consider open n-qubit quantum system described by Eq. (8) as a quantum computer with pure states. In general, we can consider open and dissipative quantum systems as a quantum computer with mixed states [44–46]. A mixed state (operator of density matrix) of n two-level quantum systems (open or closed n-qubit system) is an element of the 4^n-dimensional operator Hilbert space (Liouville space). It allows one to use a quantum computer model with four-valued logic [44–46]. The quantum gates of this model are real, completely positive, trace-preserving superoperators that act on mixed state [44,46]. Bifurcations of pure quantum states can be used for quantum gate control.

ACKNOWLEDGMENT

This work was partially supported by the RFBR Grant No. 02-02-16444.

APPENDIX

For the concept of Liouville space and superoperators see Refs. [20–37].

Liouville space

The space of linear operators acting on a Hilbert space $\mathcal{H}$ is a complex linear space $\overline{\mathcal{H}}$. We denote an element A of $\overline{\mathcal{H}}$ by a ket-vector $|A)$. The inner product of two elements $|A)$ and $|B)$ of $\overline{\mathcal{H}}$ is defined as $(A|B)=Tr(A^\dagger B)$. The norm $\|A\|=\sqrt{(A|A)}$ is the Hilbert-Schmidt norm of operator A. A new Hilbert space $\overline{\mathcal{H}}$ with the inner product is called Liouville space attached to $\mathcal{H}$ or the associated Hilbert space, or Hilbert-Schmidt space [20–37].

Let $\{|x\rangle\}$ be an orthonormal basis of $\mathcal{H}$:

$$\langle x|x'\rangle=\delta(x-x'),\quad \int dx|x\rangle\langle x|=I.$$

Then $|x,x')=||x\rangle\langle x'|)$ is an orthonormal basis of the Liouville space $\overline{\mathcal{H}}$:

$$(x,x'|y,y')=\delta(x-x')\,\delta(y-y'),$$

$$\int dx\int dx'|x,x')(x,x'|=\hat{I}. \tag{A1}$$

For an arbitrary element $|A)$ of $\overline{\mathcal{H}}$ we have

$$|A)=\int dx\int dx'|x,x')(x,x'|A), \tag{A2}$$

where

$$(x,x'|A)=\mathrm{tr}((|x\rangle\langle x'|)^\dagger A)=\mathrm{tr}(|x'\rangle\langle x|A)=\langle x|A|x'\rangle$$
$$=A(x,x')$$

is a kernel of the operator A. An operator ρ of density matrix ($\mathrm{tr}\rho=1,\rho^\dagger=\rho,\rho\geqslant 0$) can be considered as an element $|\rho)$ of the Liouville space $\overline{\mathcal{H}}$. Using Eq. (A2), we get

$$|\rho) = \int dx \int dx' |x,x')(x,x'|\rho), \tag{A3}$$

where the trace is represented by

$$(I|\rho) = \mathrm{tr}\rho = \int dx \quad (x,x|\rho) = 1.$$

Superoperators

Operators that act on $\overline{\mathcal{H}}$ are called superoperators and we denote them, in general, by a hat.

For an arbitrary superoperator $\hat{\Lambda}$ on $\overline{\mathcal{H}}$, which is defined by

$$\hat{\Lambda}|A) = |\hat{\Lambda}(A)),$$

we have

$$(x,x'|\hat{\Lambda}|A) = \int dy \int dy' (x,x'|\hat{\Lambda}|y,y')(y,y'|A)$$
$$= \int dy \int dy' \Lambda(x,x',y,y')A(y,y'),$$

where $\Lambda(x,x',y,y')$ is a kernel of the superoperator $\hat{\Lambda}$.

Let A be a linear operator in the Hilbert space $\mathcal{H}$. We can define the multiplication superoperators $\hat{L}_A$ and $\hat{R}_A$ by the following equations:

$$\hat{L}_A|B) = |AB), \quad \hat{R}_A|B) = |BA).$$

The superoperator kernels can be easily derived. For example, in the basis $|x,x')$ we have

$$(x,x'|\hat{L}_A|B) = \int dy \int dy' (x,x'|\hat{L}_A|y,y')(y,y'|B)$$
$$= \int dy \int dy' L_A(x,x',y,y')B(y,y').$$

Using

$$(x,x'|AB) = \langle x|AB|x'\rangle = \int dy \int dy' \langle x|A|y\rangle\langle y|B|y'\rangle \times \langle y'|x'\rangle,$$

we get kernel of the left multiplication superoperator

$$(\hat{L}_A)(x,x',y,y') = \langle x|A|y\rangle\langle x'|y'\rangle = A(x,y)\,\delta(x'-y').$$

Left superoperators $\hat{L}_A^{\pm}$ are defined as Lie and Jordan multiplication by

$$\hat{L}_A^- B = \frac{1}{i\hbar}(AB-BA), \quad \hat{L}_A^+ B = \frac{1}{2}(AB+BA).$$

The left superoperators $\hat{L}_A^{\pm}$ and right superoperators $\hat{R}_A^{\pm}$ are connected by

$$\hat{L}_A^- = -\hat{R}_A^-, \quad \hat{L}_A^+ = \hat{R}_A^+.$$

An algebra of the superoperators $\hat{L}_A^{\pm}$ is defined [8] by the following relations:

(1) Lie relations

$$\hat{L}_{A\cdot B}^- = \hat{L}_A^- \hat{L}_B^- - \hat{L}_B^- \hat{L}_A^-.$$

(2) Jordan relations

$$\hat{L}^+_{(A\circ B)\circ C} + \hat{L}_B^+ \hat{L}_C^+ \hat{L}_A^+ + \hat{L}_A^+ \hat{L}_C^+ \hat{L}_B^+ = \hat{L}^+_{A\circ B}\hat{L}_C^+ + \hat{L}^+_{B\circ C}\hat{L}_A^+ + \hat{L}^+_{A\circ C}\hat{L}_B^+,$$

$$\hat{L}^+_{(A\circ B)\circ C} + \hat{L}_B^+ \hat{L}_C^+ \hat{L}_A^+ + \hat{L}_A^+ \hat{L}_C^+ \hat{L}_B^+ = \hat{L}_C^+ \hat{L}^+_{A\circ B} + \hat{L}_B^+ \hat{L}^+_{A\circ C} + \hat{L}_A^+ \hat{L}^+_{B\circ C},$$

$$\hat{L}_C^+ \hat{L}^+_{A\circ B} + \hat{L}_B^+ \hat{L}^+_{A\circ C} + \hat{L}_A^+ \hat{L}^+_{B\circ C} = \hat{L}^+_{A\circ B}\hat{L}_C^+ + \hat{L}^+_{B\circ C}\hat{L}_A^+ + \hat{L}^+_{A\circ C}\hat{L}_B^+.$$

(3) Mixed relations

$$\hat{L}^+_{A\cdot B} = \hat{L}_A^- \hat{L}_B^+ - \hat{L}_B^+ \hat{L}_A^-,$$

$$\hat{L}^-_{A\circ B} = \hat{L}_A^+ \hat{L}_B^- + \hat{L}_B^+ \hat{L}_A^-,$$

$$\hat{L}^+_{A\circ B} = \hat{L}_A^+ \hat{L}_B^+ - \frac{\hbar^2}{4}\hat{L}_B^- \hat{L}_A^-,$$

$$\hat{L}_B^+ \hat{L}_A^+ - \hat{L}_A^+ \hat{L}_B^+ = -\frac{\hbar^2}{4}\hat{L}^-_{A\cdot B},$$

where

$$A\cdot B = \frac{1}{i\hbar}(AB-BA), \quad A\circ B = \frac{1}{2}(AB+BA).$$

[1] V. S. Maskevich, *Laser Kinetics* (Elsevier, Amsterdam, 1967).
[2] V.E. Tarasov, Phys. Lett. B **323**, 296 (1994).
[3] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976).
[4] I. Prigogine, *From Being to Becoming* (Freeman, San Francisco, 1980).
[5] H. Haken, *Advanced Synergetics: Instability Hierarchies of Self-Organizing Systems and Device* (Springer-Verlag, Berlin, 1983).
[6] O. E. Lanford in *Hydrodynamic Instabilities and the Transition to Turbulence* edited by H. L. Swinney and J. P. Gollub (Springer-Verlag, Berlin, 1981), Chap. 2.
[7] V.E. Tarasov, Phys. Lett. A **288**, 173 (2001).
[8] V.E. Tarasov, Moscow Univ. Phys. Bull. **6**, 6 (2001).
[9] V. E. Tarasov, preprint SINP MSU 2001-22/662 (unpublished).
[10] E.P. Davies, Commun. Math. Phys. **19**, 83 (1970).

[11] G. Lindblad, Rep. Math. Phys. **10**, 393 (1976).
[12] H. Spohn, Rep. Math. Phys. **10**, 189 (1976).
[13] H. Spohn, Lett. Math. Phys. **2**, 33 (1977).
[14] H. Spohn and J.L. Lebowitz, Commun. Math. Phys. **54**, 97 (1977).
[15] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976)
[16] C. Anastopoulos and J.J. Halliwell, Phys. Rev. D **51**, 6870 (1995).
[17] A. Isar, A. Sandulescu, and W. Scheid, LANL e-print quant-ph/9605041
[18] T. Dittrich and R. Craham, Europhys. Lett. **4**, 263 (1987).
[19] J.M.T. Thompson and T.S. Lunn, Appl. Math. Model. **5**, 143 (1981).
[20] J.A. Crawford, Nuovo Cimento **10**, 698 (1958).
[21] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).
[22] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **46**, 257 (1960).
[23] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **46**, 570 (1960).
[24] H. Primas, Helv. Phys. Acta **34**, 331 (1961).
[25] G. Emch, Helv. Phys. Acta **37**, 532 (1964).
[26] G. Sauermann, Physica (Amsterdam) **32**, 2017 (1966).
[27] H. Gabriel, Phys. Rev. **181**, 506 (1969).
[28] J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970), Chaps. 2.11 and 2.12.
[29] M. Schmutz, Z. Phys. B **30**, 97 (1978).
[30] K. Blum, *Density Matrix Theory and Applications* (Plenum Press, New York, 1981), Chap. 7.6.
[31] A. Abragam and M. Goldman, *Nuclear Magnetism. Order and Disorder* (Clarendon Press, Oxford, 1982), Chap. 1B.
[32] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **77**, 32 (1987).
[33] E. Fick and G. Sauermann, *The Quantum Statistics of Dynamic Processes* (Springer-Verlag, Berlin, 1990), Chaps. 3.1, 8.1, 8.2.
[34] M. Suzuki, Int. J. Mod. Phys. B **5**, 1821 (1991).
[35] V. Chernyak and S. Mukamel, Phys. Rev. **52**, 3601 (1995).
[36] V. E. Tarasov, *Quantum Mechanics: Lectures on Theory Foundation* (Vuzovskaya Kniga, Moscow, 2000), Chaps. 4,10,11,15.
[37] V. E. Tarasov, *Mathematical Introduction to Quantum Mechanics* (MAI, Moscow, 2000), Chaps. 2,3,5.
[38] V.E. Tarasov, Theor. Math. Phys. **110**, 73 (1997).
[39] J. Marsden and M. McCracken, *The Hopf Bifurcation and its Applications* (Springer, Berlin, 1976).
[40] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[41] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer-Verlag, Berlin, 1987).
[42] H. Haken, *Laser Light Dynamics* (Elsevier, Amsterdam, 1985), Chaps. 10 and 11.
[43] A. Sandulescu and H. Scutaru, Ann. Phys. (N.Y.) **173**, 277 (1987).
[44] V.E. Tarasov, J. Phys. A **35**, 5207 (2002).
[45] V. E. Tarasov, Preprint SINP MSU. 2001-31/671; LANL e-print quant-ph/0112025
[46] V.E. Tarasov, LANL e-print quant-ph/0201033