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Quantum computer with mixed states and four-valued logic

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Abstract

In this paper we discuss a model of quantum computer in which a state is an operator of density matrix and gates are general quantum operations, not necessarily unitary. A mixed state (operator of density matrix) of n two-level quantum systems is considered as an element of 4^n-dimensional operator Hilbert space (Liouville space). It allows us to use a quantum computer model with four-valued logic. The gates of this model are general superoperators which act on n-ququat state. Ququat is a quantum state in a four-dimensional (operator) Hilbert space. Unitary two-valued logic gates and quantum operations for an n-qubit open system are considered as four-valued logic gates acting on n-ququats. We discuss properties of quantum four-valued logic gates. In the paper we study universality for quantum four-valued logic gates.

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1. Introduction

The usual models of a quantum computer deal only with unitary gates on pure states. In these models it is difficult or impossible to deal formally with measurements, dissipation, decoherence and noise. It turns out that the restriction to pure states and unitary gates is unnecessary [1–3]. Understanding the dynamics of open systems is important for studying quantum noise processes [4–6], quantum error correction [7–11], decoherence effects in quantum computations [12–17] and performing simulations of open quantum systems [18–23].

In this paper we generalize the usual model of a quantum computer to a model in which a state is a density matrix operator and gates are general superoperators (quantum operations), not necessarily unitary. The pure state of n two-level closed quantum systems is an element of 2^n-dimensional Hilbert space and it allows us to consider a quantum computer model with two-valued logic. The gates of this computer model are unitary operators which act on a such state. In the general case, the mixed state (operator of density matrix) of n two-level quantum systems is an element of 4^n-dimensional operator Hilbert space (Liouville space). It allows us to use a quantum computer model with four-valued logic. The gates of this model are general superoperators (quantum operations) which act on general n-ququat state. A ququat [2, 3] is a quantum state in a four-dimensional (operator) Hilbert space. Unitary gates and quantum operations for a quantum two-valued logic computer can be considered as four-valued logic gates of the new model. In the paper we consider universality for general quantum four-valued logic gates acting on ququats.

In sections 2 and 3 the physical and mathematical backgrounds (pure and mixed states, Liouville space and superoperators) are considered. In section 4, we introduce a generalized computational basis and generalized computational states for 4^n-dimensional operator Hilbert space (Liouville space). In section 5, we study some properties of general four-valued logic gates. Unitary gates and quantum operations of a two-valued logic computer are considered as four-valued logic gates. In section 6, we introduce a four-valued classical logic formalism. We realize classical four-valued logic gates by quantum gates. In section 7, we study universality for quantum four-valued logic gates. Finally, a short conclusion is given in section 8.

2. Quantum state and qubit

2.1. Pure states

A quantum system in a pure state is described by a unit vector in a Hilbert space $\mathcal{H}$. In the Dirac notation a pure state is denoted by $|\Psi\rangle$. The Hilbert space $\mathcal{H}$ is a linear space with an inner product. The inner product for $|\Psi_1\rangle$, $|\Psi_2\rangle \in \mathcal{H}$ is denoted by $\langle\Psi_1|\Psi_2\rangle$. A quantum bit or qubit, the fundamental concept of quantum computations, is a two-state quantum system. The two basis states labelled $|0\rangle$ and $|1\rangle$ are orthogonal unit vectors, i.e.

$$\langle k|l\rangle = \delta_{kl}$$

where $k, l \in \{0, 1\}$. The Hilbert space of the qubit is $\mathcal{H}_2 = \mathbb{C}^2$. The quantum system which corresponds to a quantum computer (quantum circuits) consists of n quantum two-state particles. The Hilbert space $\mathcal{H}^{(n)}$ of such a system is a tensor product of n Hilbert spaces $\mathcal{H}_2$ of one two-state particle: $\mathcal{H}^{(n)} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2$. The space $\mathcal{H}^{(n)}$ is a 2^n-dimensional complex linear space. Let us choose a basis for $\mathcal{H}^{(n)}$ which consists of the 2^n orthonormal states $|k\rangle$, where k is in binary representation. The state $|k\rangle$ is a tensor product of states $|k_i\rangle$ in $\mathcal{H}^{(n)}$:

$$|k\rangle = |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle = |k_1 k_2 \cdots k_n\rangle$$

where $k_i \in \{0, 1\}$, $i = 1, 2, \ldots, n$. This basis is usually called the computational basis which has 2^n elements. A pure state $|\Psi(t)\rangle \in \mathcal{H}^{(n)}$ is generally a superposition of the basis states

$$|\Psi(t)\rangle = \sum_{k=0}^{2^n-1} a_k(t)|k\rangle \tag{1}$$

where $\sum_{k=0}^{2^n-1} |a_k(t)|^2 = 1$.

2.2. Mixed states

In general, a quantum system is not in a pure state. Open quantum systems are not really isolated from the rest of the universe, so it does not have a well-defined pure state. Landau and von Neumann introduced a mixed state and a density matrix into quantum theory. A density matrix is a Hermitian ($\rho^\dagger = \rho$), positive ($\rho > 0$) operator on $\mathcal{H}^{(n)}$ with unit trace ($\mathrm{Tr}\,\rho = 1$). Pure states can be characterized by idempotent condition $\rho^2 = \rho$. A pure state (1) can be represented by the operator $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$.

One can represent an arbitrary density matrix operator $\rho(t)$ for n-qubits in terms of tensor products of Pauli matrices σ_μ:

$$\rho(t) = \frac{1}{2^n} \sum_{\mu_1 \cdots \mu_n} P_{\mu_1 \cdots \mu_n}(t) \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n} \tag{2}$$

where $\mu_i \in \{0, 1, 2, 3\}$, $i = 1, \ldots, n$. Here σ_μ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

The real expansion coefficients $P_{\mu_1 \ldots \mu_n}(t)$ are given by

$$P_{\mu_1 \cdots \mu_n}(t) = \mathrm{Tr}\left(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n} \rho(t)\right).$$

Normalization ($\mathrm{Tr}\,\rho = 1$) requires that $P_{0 \cdots 0}(t) = 1$. Since the eigenvalues of the Pauli matrices are ± 1, the expansion coefficients satisfy $\left|P_{\mu_1 \cdots \mu_n}(t)\right| \leqslant 1$.

3. Liouville space and superoperators

For the concept of Liouville space and superoperators, see [24–38].

3.1. Liouville space

The space of linear operators acting on a 2^n-dimensional Hilbert space $\mathcal{H}^{(n)}$ is a $(2^n)^2 = 4^n$-dimensional complex linear space $\overline{\mathcal{H}}^{(n)}$. We denote an element A of $\overline{\mathcal{H}}^{(n)}$ by a ket-vector $|A)$. The inner product of two elements $|A)$ and $|B)$ of $\overline{\mathcal{H}}^{(n)}$ is defined as

$$(A|B) = \mathrm{Tr}(A^\dagger B). \tag{5}$$

The norm $\|A\| = \sqrt{(A|A)}$ is the Hilbert–Schmidt norm of operator A. A new Hilbert space $\overline{\mathcal{H}}^{(n)}$ with scalar product (5) is called the Liouville space attached to $\mathcal{H}^{(n)}$ or the associated Hilbert space, or Hilbert–Schmidt space [24–38].

Let $\{|k\rangle\}$ be an orthonormal basis of $\mathcal{H}^{(n)}$:

$$\langle k|k'\rangle = \delta_{kk'} \qquad \sum_{k=0}^{2^n-1} |k\rangle\langle k| = I.$$

Then $|k, l) = \| k\rangle\langle l|)$ is an orthonormal basis of the Liouville space $\overline{\mathcal{H}}^{(n)}$:

$$(k, l|k', l') = \delta_{kk'}\delta_{ll'} \qquad \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} |k, l)(k, l| = \hat{I}. \tag{6}$$

This operator basis has 4^n elements. Note that

$$|k, l) = |k_1, l_1) \otimes |k_2, l_2) \otimes \cdots \otimes |k_n, l_n) \tag{7}$$

where $k_i, l_i \in \{0, 1\}$, $i = 1, \ldots, n$, and

$$|k_i, l_i) \otimes |k_j, l_j) = ||k_i\rangle \otimes |k_j\rangle, \langle l_i| \otimes \langle l_j|).$$

For an arbitrary element $|A)$ of $\overline{\mathcal{H}}^{(n)}$ we have

$$|A) = \sum_{k=0}^{2^n-1}\sum_{l=0}^{2^n-1} |k,l)(k,l|A) \tag{8}$$

with

$$(k,l|A) = \mathrm{Tr}(|l\rangle\langle k|A) = \langle k|A|l\rangle = A_{kl}.$$

3.2. Superoperators

Operators which act on $\overline{\mathcal{H}}^{(n)}$ are called superoperators and we denote them in general by the hat.

For an arbitrary superoperator $\hat{\mathcal{E}}$ on $\overline{\mathcal{H}}^{(n)}$ we have

$$\begin{aligned}(k,l|\hat{\mathcal{E}}|A) &= \sum_{k'=0}^{2^n-1}\sum_{l'=0}^{2^n-1}(k,l|\hat{\mathcal{E}}|k',l')(k',l'|A)\\ &= \sum_{k'=0}^{2^n-1}\sum_{l'=0}^{2^n-1}\mathcal{E}_{klk'l'}A_{k'l'}.\end{aligned}$$

Let A be a linear operator in Hilbert space $\mathcal{H}^{(n)}$. Then the superoperators $\hat{L}_A$ and $\hat{R}_A$ will be defined by

$$\hat{L}_A|B) = |AB) \qquad \hat{R}_A|B) = |BA). \tag{9}$$

The superoperator $\hat{\mathcal{P}} = |A)(B|$ is defined by

$$\hat{\mathcal{P}}|C) = |A)(B|C) = |A)\,\mathrm{Tr}(B^\dagger C). \tag{10}$$

The superoperator $\hat{\mathcal{E}}^\dagger$ is called the adjoint superoperator for $\hat{\mathcal{E}}$ if

$$(\hat{\mathcal{E}}^\dagger(A)|B) = (A|\hat{\mathcal{E}}(B)) \tag{11}$$

for all $|A)$ and $|B)$ from $\overline{\mathcal{H}}^{(n)}$. For example, if $\hat{\mathcal{E}} = \hat{L}_A\hat{R}_B$, then $\hat{\mathcal{E}}^\dagger = \hat{L}_{A^\dagger}\hat{R}_{B^\dagger}$. If $\hat{\mathcal{E}} = \hat{L}_A$, then $\hat{\mathcal{E}}^\dagger = \hat{L}_{A^\dagger}$.

The superoperator $\hat{\mathcal{E}}$ is called unital if $\hat{\mathcal{E}}|I) = |I)$.

4. Generalized computational basis and ququats

Let us introduce a generalized computational basis and generalized computational states for 4^n-dimensional operator Hilbert space (Liouville space).

4.1. Pauli representation

Pauli matrices (3) and (4) can be considered as a basis in operator space. Let us write the Pauli matrices (3) and (4) in the form

$$\begin{aligned}\sigma_1 &= |0\rangle\langle 1| + |1\rangle\langle 0| = |0,1) + |1,0)\\ \sigma_2 &= -\mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0| = -\mathrm{i}(|0,1) - |1,0))\\ \sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1| = |0,0) - |1,1)\\ \sigma_0 &= I = |0\rangle\langle 0| + |1\rangle\langle 1| = |0,0) + |1,1).\end{aligned}$$

Let us use the formulae

$$|0,0) = \tfrac{1}{2}(|\sigma_0) + |\sigma_3)) \qquad |1,1) = \tfrac{1}{2}(|\sigma_0) - |\sigma_3))$$

$$|0,1) = \tfrac{1}{2}(|\sigma_1) + \mathrm{i}|\sigma_2)) \qquad |1,0) = \tfrac{1}{2}(|\sigma_1) - \mathrm{i}|\sigma_2)).$$

It allows us to rewrite the operator basis

$$|k,l) = |k_1,l_1) \otimes |k_2,l_2) \otimes \cdots \otimes |k_n,l_n)$$

by complete basis operators

$$|\sigma_\mu) = \big|\sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \cdots \otimes \sigma_{\mu_n}\big)$$

where $\mu_i = 2k_i + l_i$, i.e. $\mu_i \in \{0,1,2,3\}$, $i = 1,\ldots,n$. The basis $|\sigma_\mu)$ is orthogonal

$$(\sigma_\mu|\sigma_{\mu'}) = 2^n \delta_{\mu\mu'}$$

and complete operator basis

$$\frac{1}{2^n}\sum_{\mu}^{N-1} |\sigma_\mu)(\sigma_\mu| = \hat{I}$$

where $N = 4^n$. For an arbitrary element $|A)$ of $\overline{\mathcal{H}}^{(n)}$ we have the Pauli representation by

$$|A) = \frac{1}{2^n}\sum_{\mu=0}^{N-1} |\sigma_\mu)(\sigma_\mu|A) \tag{12}$$

with the complex coefficients $(\sigma_\mu|A) = \mathrm{Tr}(\sigma_\mu A)$. We can rewrite formula (2) using the complete operator basis $|\sigma_\mu)$ in Liouville space $\overline{\mathcal{H}}^{(n)}$:

$$|\rho(t)) = \frac{1}{2^n}\sum_{\mu=0}^{N-1} |\sigma_\mu) P_\mu(t) \tag{13}$$

where $\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$, $\mu = (\mu_1 \cdots \mu_n)$, $N = 4^n$ and $P_\mu(t) = (\sigma_\mu|\rho(t))$.

The density matrix operator $\rho(t)$ is a self-adjoint operator with unit trace. It follows that

$$P^*_\mu(t) = P_\mu(t) \qquad P_0(t) = (\sigma_0|\rho(t)) = 1.$$

In the general case,

$$\frac{1}{2^n}\sum_{\mu=0}^{N-1} P^2_\mu(t) = (\rho(t)|\rho(t)) = \mathrm{Tr}(\rho^2(t)) \leqslant 1. \tag{14}$$

Note that the Schwarz inequality

$$|(A|B)|^2 \leqslant (A|A)(B|B)$$

leads to

$$|(I|\rho(t))|^2 \leqslant (I|I)(\rho(t)|\rho(t)).$$

We rewrite this inequality in the form

$$1 = |\mathrm{Tr}\,\rho(t)|^2 \leqslant 2^n\ \mathrm{Tr}(\rho^2(t)) = \sum_{\mu=0}^{N-1} P^2_\mu(t) \tag{15}$$

where $N = 4^n$. Using (14) and (15) we have

$$\frac{1}{\sqrt{2^n}} \leqslant \mathrm{Tr}(\rho^2(t)) \leqslant 1 \qquad \text{or} \qquad 1 \leqslant \sum_{\mu=0}^{N-1} P^2_\mu(t) \leqslant 2^n. \tag{16}$$

4.2. Generalized computational basis

Let us define the orthonormal basis of Liouville space. In the general case, the state $\rho(t)$ of the n-qubit system is an element of Hilbert space $\overline{\mathcal{H}}^{(n)}$. The basis for $\overline{\mathcal{H}}^{(n)}$ consists of the $2^{2n} = 4^n$ orthonormal basis elements denoted by $|\mu)$.

Definition. *The basis of Liouville space* $\overline{\mathcal{H}}^{(n)}$ *is defined by*

$$|\mu) = |\mu_1 \cdots \mu_n) = \frac{1}{\sqrt{2^n}}|\sigma_\mu) = \frac{1}{\sqrt{2^n}}\big|\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}\big) \tag{17}$$

where $N = 4^n$, $\mu_i \in \{0, 1, 2, 3\}$ *and*

$$(\mu|\mu') = \delta_{\mu\mu'} \qquad \sum_{\mu=0}^{N-1} |\mu)(\mu| = \hat{I} \tag{18}$$

is called the 'generalized computational basis'.

Here μ is a four-valued representation of

$$\mu = \mu_1 4^{n-1} + \cdots + \mu_{n-1}4 + \mu_n. \tag{19}$$

The pure state of n two-level closed quantum systems is an element of 2^n-dimensional functional Hilbert space $\mathcal{H}^{(n)}$. It leads to a quantum computer model with two-valued logic. In the general case, the mixed state $\rho(t)$ of n two-level (open or closed) quantum systems is an element of 4^n-dimensional operator Hilbert space $\overline{\mathcal{H}}^{(n)}$ (Liouville space). It leads to a four-valued logic model for the quantum computer.

The state $|\rho(t))$ of the quantum computation at any point in time is a superposition of basis elements,

$$|\rho(t)) = \sum_{\mu=0}^{N-1} |\mu)\rho_\mu(t) \tag{20}$$

where $\rho_\mu(t) = (\mu|\rho(t))$ are real numbers (functions) satisfying normalized condition

$$\rho_0(t) = \frac{1}{\sqrt{2^n}}(\sigma_0|\rho(t)) = \frac{1}{\sqrt{2^n}}\,\mathrm{Tr}(\rho(t)) = \frac{1}{\sqrt{2^n}}. \tag{21}$$

4.3. Generalized computational states

Generalized computational basis elements $|\mu)$ are not quantum states for $\mu \neq 0$. It follows from normalized condition (21). The general quantum state in the Pauli representation has the form (20). Let us define simple computational quantum states.

Definition. *Quantum states in Liouville space defined by*

$$|\mu] = \frac{1}{2^n}(|\sigma_0) + |\sigma_\mu)(1 - \delta_{\mu 0})) \tag{22}$$

or

$$|\mu] = \frac{1}{\sqrt{2^n}}(|0) + |\mu)(1 - \delta_{\mu 0})). \tag{23}$$

are called 'generalized computational states'.

Note that all states $|\mu]$, where $\mu \neq 0$, are pure states, since $[\mu|\mu] = 1$. The state $|0]$ is a maximally mixed state. The states $|\mu]$ are elements of Liouville space $\overline{\mathcal{H}}^{(n)}$.

The quantum state in a four-dimensional Hilbert space is usually called ququat, qu-quart [39] or qudit [40–44] with $d = 4$. Usually the ququat is considered as a four-level quantum system. We consider the ququat as a general quantum state in a four-dimensional operator Hilbert space.

Definition. *A quantum state in four-dimensional operator Hilbert space (Liouville space) $\overline{\mathcal{H}}^{(1)}$ associated with a single qubit of space $\mathcal{H}^{(1)} = \mathcal{H}_2$ is called a 'single ququat'. A quantum state in 4^n-dimensional Liouville space $\overline{\mathcal{H}}^{(n)}$ associated with an n-qubits system is called an 'n-ququat'.*

Example. For the single ququat the states $|\mu]$ are

$$|0] = \tfrac{1}{2}|\sigma_0) \qquad |k] = \tfrac{1}{2}(|\sigma_0) + |\sigma_k))$$

or

$$|0] = \frac{1}{\sqrt{2}}|0) \qquad |k] = \frac{1}{\sqrt{2}}(|0) + |k)).$$

It is convenient to use matrices for quantum states. In matrix representation the single ququat computational basis $|\mu)$ can be represented by

$$|0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |1) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |3) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

In this representation single ququat generalized computational states $|\mu]$ are represented by

$$|0] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |1] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \qquad |2] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad |3] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

A general single ququat quantum state $|\rho) = \sum_{\mu=0}^{3} |\mu)\rho_\mu$ is represented by

$$|\rho) = \begin{pmatrix}\rho_0\\\rho_1\\\rho_2\\\rho_3\end{pmatrix}$$

where $\rho_0 = 1/\sqrt{2}$ and $\rho_1^2 + \rho_2^2 + \rho_3^2 \leqslant \sqrt{2}$.

We can use the other matrix representation for the states $|\rho]$ which have no coefficient $1/\sqrt{2^n}$. In this representation single ququat generalized computational states $|\mu]$ are represented by

$$|0] = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad |1] = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad |2] = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \qquad |3] = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

A general single ququat quantum state

$$|\rho] = \begin{bmatrix}1\\P_1\\P_2\\P_3\end{bmatrix}$$

where $P_1^2 + P_2^2 + P_3^2 \leqslant 1$, is a superposition of generalized computational states

$$|\rho] = |0](1 - P_1 - P_2 - P_3) + |1]P_1 + |2]P_2 + |3]P_3.$$

5. Quantum four-valued logic gates

5.1. Superoperators and quantum gates

Unitary evolution is not the most general type of state change possible for quantum systems. The most general state change of a quantum system is a positive map which is called a quantum operation or superoperator. For the concept of quantum operations, see [5, 45–48].

Quantum operations can be considered as generalized quantum gates acting on general (mixed) states. Let us define a quantum four-valued logic gate.

Definition. *A quantum four-valued logic gate is a superoperator $\hat{\mathcal{E}}$ on Liouville space $\overline{\mathcal{H}}^{(n)}$ which maps a density matrix operator $|\rho)$ of n-ququats to a density matrix operator $|\rho')$ of n-ququats.*

Let us consider a superoperator $\hat{\mathcal{E}}$ which maps density matrix operator $|\rho)$ to density matrix operator $|\rho')$. If $|\rho)$ is an density matrix operator, then $\hat{\mathcal{E}}|\rho)$ should also be a density matrix operator. Therefore we have some requirements for superoperator $\hat{\mathcal{E}}$. The requirements for a superoperator $\hat{\mathcal{E}}$ on the space $\mathcal{H}^{(n)}$ to be the quantum four-valued logic gate are as follows:

1. The superoperator $\hat{\mathcal{E}}$ is a *real* superoperator, i.e. $(\hat{\mathcal{E}}(A))^\dagger = \hat{\mathcal{E}}(A^\dagger)$ for all A or $(\hat{\mathcal{E}}(\rho))^\dagger = \hat{\mathcal{E}}(\rho)$. The real superoperator $\hat{\mathcal{E}}$ maps self-adjoint operator ρ to self-adjoint operator $\hat{\mathcal{E}}(\rho)$: $(\hat{\mathcal{E}}(\rho))^\dagger = \hat{\mathcal{E}}(\rho)$.
2. The superoperator $\hat{\mathcal{E}}$ is a *positive* superoperator, i.e. $\hat{\mathcal{E}}$ maps positive operators to positive operators: $\hat{\mathcal{E}}(A^2) > 0$ for all $A \neq 0$ or $\hat{\mathcal{E}}(\rho) \geqslant 0$.
   We have to assume the superoperator $\hat{\mathcal{E}}$ to be not merely positive but completely positive. The superoperator $\hat{\mathcal{E}}$ is a *completely positive* map of Liouville space, i.e. the positivity remains if we extend the Liouville space $\overline{\mathcal{H}}^{(n)}$ by adding more qubits. That is, the superoperator $\hat{\mathcal{E}} \otimes \hat{I}^{(m)}$ must be positive, where $\hat{I}^{(m)}$ is the identity superoperator on some Liouville space $\overline{\mathcal{H}}^{(m)}$.
3. The superoperator $\hat{\mathcal{E}}$ is a *trace-preserving* map, i.e.
$$(I|\hat{\mathcal{E}}|\rho) = (\hat{\mathcal{E}}^\dagger(I)|\rho) = 1 \qquad \text{or} \qquad \hat{\mathcal{E}}^\dagger(I) = I. \tag{24}$$
   As the result we have the following definition.

Definition. *A quantum four-valued logic gate is a real positive (or completely positive) trace-preserving superoperator $\hat{\mathcal{E}}$ on Liouville space $\overline{\mathcal{H}}^{(n)}$.*

In the general case, we can consider linear and nonlinear quantum four-valued logic gates. Let the superoperator $\hat{\mathcal{E}}$ be a *convex linear* map on the set of density matrix operators, i.e.

$$\hat{\mathcal{E}}\left(\sum_s \lambda_s \rho_s\right) = \sum_s \lambda_s \hat{\mathcal{E}}(\rho_s)$$

where all λ_s are $0 < \lambda_s < 1$ and $\sum_s \lambda_s = 1$. Any convex linear map of density matrix operators can be uniquely extended to a *linear* map on Hermitian operators. Note that any linear completely positive superoperator can be represented by

$$\hat{\mathcal{E}} = \sum_{j=1}^{m} \hat{L}_{A_j} \hat{R}_{A_j^\dagger}.$$

If this is a trace-preserving superoperator, then

$$\sum_{j=1}^{m} A_j^{\dagger} A_j = I$$

i.e. condition (24) is satisfied.

The restriction to linear gates is unnecessary. Let us consider a linear real completely positive superoperator $\hat{\mathcal{E}}$ which is not trace-preserving. This superoperator is not a quantum gate. Let $(I|\hat{\mathcal{E}}|\rho) = \mathrm{Tr}(\hat{\mathcal{E}}(\rho))$ be the probability that the process represented by the superoperator $\hat{\mathcal{E}}$ occurs. Since the probability is non-negative and never exceeds 1, it follows that the superoperator $\hat{\mathcal{E}}$ is a trace-decreasing superoperator:

$$0 \leqslant (I|\hat{\mathcal{E}}|\rho) \leqslant 1 \qquad \text{or} \qquad \hat{\mathcal{E}}^{\dagger}(I) \leqslant I.$$

In general, any linear real completely positive trace-decreasing superoperator generates a quantum four-valued logic gate. The quantum four-valued logic gate can be defined as *nonlinear* superoperator $\hat{\mathcal{N}}$ by

$$\hat{\mathcal{N}}|\rho) = (I|\hat{\mathcal{E}}|\rho)^{-1}\hat{\mathcal{E}}|\rho) \qquad \text{or} \qquad \hat{\mathcal{N}}(\rho) = \frac{\hat{\mathcal{E}}(\rho)}{\mathrm{Tr}(\hat{\mathcal{E}}(\rho))} \tag{25}$$

where $\hat{\mathcal{E}}$ is a linear real completely positive trace-decreasing superoperator.

In the generalized computational basis the gate $\hat{\mathcal{E}}$ can be represented by

$$\hat{\mathcal{E}} = \frac{1}{2^n} \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu} |\sigma_\mu)(\sigma_\nu| \tag{26}$$

where $N = 4^n$, μ and ν are four-valued representations of

$$\mu = \mu_1 4^{N-1} + \cdots + \mu_{N-1} 4 + \mu_N$$
$$\nu = \nu_1 4^{N-1} + \cdots + \nu_{N-1} 4 + \nu_N$$
$$\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$$

$\mu_i, \nu_i \in \{0, 1, 2, 3\}$ and $\mathcal{E}_{\mu\nu}$ are elements of some matrix.

5.2. General quantum operation as four-valued logic gates

Unitary gates and quantum operations for a quantum computer with pure states and two-valued logic can be considered as quantum four-valued logic gates acting on mixed states.

Proposition 1. *In the generalized computational basis $|\mu)$ any linear quantum operation $\hat{\mathcal{E}}$ acting on n-qubit mixed (or pure) states can be represented as a quantum four-valued logic gate $\hat{\mathcal{E}}$ on n-ququats by*

$$\hat{\mathcal{E}} = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu} |\mu)(\nu| \tag{27}$$

where $N = 4^n$,

$$\mathcal{E}_{\mu\nu} = \frac{1}{2^n} \mathrm{Tr}(\sigma_\mu \hat{\mathcal{E}}(\sigma_\nu)) \tag{28}$$

and $\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$.

Proof. The state ρ' in the generalized computational basis $|\mu)$ has the form

$$|\rho') = \sum_{\mu=0}^{N-1} |\mu)\rho'_\mu$$

where $N = 4^n$ and

$$\rho'_\mu = (\mu|\rho') = \frac{1}{\sqrt{2^n}}\,\mathrm{Tr}(\sigma_\mu \rho').$$

The quantum operation $\hat{\mathcal{E}}$ defines a quantum four-valued logic gate by

$$|\rho') = \hat{\mathcal{E}}|\rho) = |\hat{\mathcal{E}}(\rho)) = \sum_{\nu=0}^{N-1} |\hat{\mathcal{E}}(\sigma_\nu))\frac{1}{\sqrt{2^n}}\rho_\nu.$$

Then

$$(\mu|\rho') = \sum_{\nu=0}^{N-1} (\sigma_\mu|\hat{\mathcal{E}}(\sigma_\nu))\frac{1}{2^n}\rho_\nu.$$

Finally, we obtain

$$\rho'_\mu = \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu}\rho_\nu$$

where

$$\mathcal{E}_{\mu\nu} = \frac{1}{2^n}\,(\sigma_\mu|\hat{\mathcal{E}}(\sigma_\nu)) = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\mu \hat{\mathcal{E}}(\sigma_\nu)).$$

This formula defines a relation between quantum operation $\hat{\mathcal{E}}$ and the real $4^n \times 4^n$ matrix $\mathcal{E}_{\mu\nu}$ of a quantum four-valued logic gate.

Quantum four-valued logic gates $\hat{\mathcal{E}}$ in the matrix representation are represented by $4^n \times 4^n$ matrices $\mathcal{E}_{\mu\nu}$. The matrix of the gate $\hat{\mathcal{E}}$ is

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{00} & \mathcal{E}_{01} & \cdots & \mathcal{E}_{0a} \\ \mathcal{E}_{10} & \mathcal{E}_{11} & \cdots & \mathcal{E}_{1a} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{a0} & \mathcal{E}_{a1} & \cdots & \mathcal{E}_{aa} \end{pmatrix}$$

where $a = N - 1 = 4^n - 1$. In matrix representation the gate $\hat{\mathcal{E}}$ maps the state $|\rho) = \sum_{\nu=0}^{N-1} |\nu)\rho_\nu$ to the state $|\rho') = \sum_{\mu=0}^{N-1} |\mu)\rho'_\mu$ by

$$\rho'_\mu = \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu}\rho_\nu \tag{29}$$

where $\rho'_0 = \rho_0 = 1/\sqrt{2^n}$. It can be written in the form

$$\begin{pmatrix} \rho'_0 \\ \rho'_1 \\ \cdots \\ \rho'_a \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{00} & \mathcal{E}_{01} & \cdots & \mathcal{E}_{0a} \\ \mathcal{E}_{10} & \mathcal{E}_{11} & \cdots & \mathcal{E}_{1a} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{a0} & \mathcal{E}_{a1} & \cdots & \mathcal{E}_{aa} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_a \end{pmatrix}.$$

Since $P_\mu = \sqrt{2^n}\rho_\mu$ and $P'_\mu = \sqrt{2^n}\rho'_\mu$, it follows that representation (29) for linear gate $\hat{\mathcal{E}}$ is equivalent to

$$P'_\mu = \sum_{\nu=0}^{N-1} \mathcal{E}_{\mu\nu}P_\nu. \tag{30}$$

It can be written in the form

$$\begin{bmatrix} P'_0 \\ P'_1 \\ \cdots \\ P'_a \end{bmatrix} = \begin{pmatrix} \mathcal{E}_{00} & \mathcal{E}_{01} & \cdots & \mathcal{E}_{0a} \\ \mathcal{E}_{10} & \mathcal{E}_{11} & \cdots & \mathcal{E}_{1a} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{a0} & \mathcal{E}_{a1} & \cdots & \mathcal{E}_{aa} \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \\ \cdots \\ P_a \end{bmatrix}$$

where $P_0 = P'_0 = 1$. Note that if we use different matrix representations of state we can use identical matrices for gate $\hat{\mathcal{E}}$. □

Proposition 2. *In the generalized computational basis $|\mu)$ the matrix $\mathcal{E}_{\mu\nu}$ of a linear quantum four-valued logic gate*

$$\hat{\mathcal{E}} = \sum_{j=1}^{m} \hat{L}_{A_j} \hat{R}_{A_j^\dagger} \tag{31}$$

is real, i.e. $\mathcal{E}^*_{\mu\nu} = \mathcal{E}_{\mu\nu}$ for all μ and ν. Any real matrix $\mathcal{E}_{\mu\nu}$ associated with linear (trace-preserving) quantum four-valued logic gate (31) has

$$\mathcal{E}_{0\nu} = \delta_{0\nu}. \tag{32}$$

Proof.

$$\mathcal{E}_{\mu\nu} = \frac{1}{2^n} \sum_{j=1}^{m} \mathrm{Tr}(\sigma_\mu A_j \sigma_\nu A_j^\dagger) = \frac{1}{2^n} \sum_{j=1}^{m} (A_j^\dagger \sigma_\mu | \sigma_\nu A_j^\dagger).$$

$$\mathcal{E}^*_{\mu\nu} = \frac{1}{2^n} \sum_{j=1}^{m} (A_j^\dagger \sigma_\mu | \sigma_\nu A_j^\dagger)^* = \frac{1}{2^n} \sum_{j=1}^{m} (\sigma_\nu A_j^\dagger | A_j^\dagger \sigma_\mu)$$

$$= \frac{1}{2^n} \sum_{j=1}^{m} \mathrm{Tr}(A_j \sigma_\nu A_j^\dagger \sigma_\mu) = \frac{1}{2^n} \sum_{j=1}^{m} \mathrm{Tr}(\sigma_\mu A_j \sigma_\nu A_j^\dagger) = \mathcal{E}_{\mu\nu}.$$

Let us consider the $\mathcal{E}_{0\nu}$ for gate (31):

$$\mathcal{E}_{0\nu} = \frac{1}{2^n} \mathrm{Tr}(\sigma_0 \mathcal{E}(\sigma_\nu)) = \frac{1}{2^n} \mathrm{Tr}(\mathcal{E}(\sigma_\nu))$$

$$= \frac{1}{2^n} \mathrm{Tr}\left(\sum_{j=1}^{m} A_j \sigma_\nu A_j^\dagger \right) = \frac{1}{2^n} \mathrm{Tr}\left(\left(\sum_{j=1}^{m} A_j^\dagger A_j \right) \sigma_\nu \right)$$

$$= \frac{1}{2^n} \mathrm{Tr}\, \sigma_\nu = \delta_{0\nu}.$$

In the general case, a linear quantum four-valued logic gate acts on $|0)$ by

$$\hat{\mathcal{E}}|0) = |0) + \sum_{k=1}^{N-1} T_k |k).$$

For example, a single ququat quantum gate acts by

$$\hat{\mathcal{E}}|0) = |0) + T_1|1) + T_2|2) + T_3|3).$$

If all T_k, where $k = 1, \ldots, N-1$, are equal to zero, then $\hat{\mathcal{E}}|0) = |0)$. The linear quantum gates with $T = 0$ conserve the maximally mixed state $|0]$ invariant. □

Definition. *A quantum four-valued logic gate $\hat{\mathcal{E}}$ is called a unital gate or gate with $T = 0$ if the maximally mixed state $|0]$ is invariant under the action of this gate: $\hat{\mathcal{E}}|0] = |0]$.*

The output state of a linear quantum four-valued logic gate $\hat{\mathcal{E}}$ is $|0]$ if and only if the input state is $|0]$. If $\hat{\mathcal{E}}|0] \neq |0]$, then $\hat{\mathcal{E}}$ is not a unital gate.

Proposition 3. *The matrix $\mathcal{E}_{\mu\nu}$ of linear real trace-preserving superoperator $\hat{\mathcal{E}}$ on n-ququats is an element of group $TGL(4^n - 1, \mathbb{R})$ which is a semidirect product of general linear group $GL(4^n - 1, \mathbb{R})$ and translation group $T(4^n - 1, \mathbb{R})$.*

Proof. This proposition follows from proposition 2. Any element (matrix $\mathcal{E}_{\mu\nu}$) of group $TGL(4^n - 1, \mathbb{R})$ can be represented by

$$\mathcal{E}(T, R) = \begin{pmatrix} 1 & 0 \\ T & R \end{pmatrix}$$

where T is a column with $4^n - 1$ elements, 0 is a line with $4^n - 1$ zero elements, and R is a real $(4^n - 1) \times (4^n - 1)$ matrix $R \in GL(4^n - 1, \mathbb{R})$. If R is orthogonal $(4^n - 1) \times (4^n - 1)$ matrix $(R^T R = I)$, then we have the motion group [62–64]. The group multiplication of elements $\mathcal{E}(T, R)$ and $\mathcal{E}(T', R')$ is defined by

$$\mathcal{E}(T, R)\mathcal{E}(T', R') = \mathcal{E}(T + RT', RR').$$

In particular, we have

$$\mathcal{E}(T, R) = \mathcal{E}(T, I)\mathcal{E}(0, R) \qquad \mathcal{E}(T, R) = \mathcal{E}(0, R)\mathcal{E}(R^{-1}T, I)$$

where I is a unit $(4^n - 1) \times (4^n - 1)$ matrix. □

Any linear real trace-preserving superoperator can be decomposed into unital superoperator and translation superoperator. It allows us to consider two types of linear trace-preserving superoperators:

(1) Unital superoperators $\hat{\mathcal{E}}^{(T=0)}$ with the matrices $\mathcal{E}(0, R)$. The n-ququat unital superoperator can be represented by

$$\hat{\mathcal{E}}^{(T=0)} = |0)(0| + \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} R_{kl} |k)(l|$$

where $N = 4^n$.

(2) Translation superoperators $\hat{\mathcal{E}}^{(T)}$ defined by matrices $\mathcal{E}(T, I)$ and

$$\hat{\mathcal{E}}^{(T)} = \sum_{\mu=0}^{N-1} |\mu)(\mu| + \sum_{k=1}^{N-1} T_k |k)(0|.$$

5.3. Decomposition for linear superoperators

Let us consider the n-ququat linear real superoperator

$$\hat{\mathcal{E}} = |0)(0| + \sum_{\mu=1}^{N-1} T_\mu |\mu)(0| + \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} R_{\mu\nu} |\mu)(\nu| \tag{33}$$

where $N = 4^n$.

Proposition 4 (Singular valued decomposition for matrix)**.** *Any real matrix R can be written in the form $R = \mathcal{U}_1 D \mathcal{U}_2^T$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are real orthogonal $(N - 1) \times (N - 1)$ matrices and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N-1})$ is a diagonal $(N - 1) \times (N - 1)$ matrix such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N-1} \geqslant 0$.*

Proof. This proposition is proved in [61, 69–71]. □

In the general case, we have the following proposition.

Proposition 5 (Singular valued decomposition for superoperator)**.** *Any linear real superoperator (33) can be represented by*

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}^{(T)} \hat{\mathcal{U}}_1 \hat{D} \hat{\mathcal{U}}_2 \tag{34}$$

where $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$ are unital orthogonal superoperators, such that

$$\hat{\mathcal{U}}_i = |0)(0| + \sum_{\mu=1}^{N-1}\sum_{\nu=1}^{N-1} \mathcal{U}^{(i)}_{\mu\nu}|\mu)(\nu| \tag{35}$$

$\hat{D}$ is a unital diagonal superoperator, such that

$$\hat{D} = |0)(0| + \sum_{\mu=1}^{N-1} \lambda_\mu |\mu)(\mu| \tag{36}$$

where $\lambda_\mu \geqslant 0$. $\hat{\mathcal{E}}^{(T)}$ is a translation superoperator, such that

$$\hat{\mathcal{E}}^{(T)} = |0)(0| + \sum_{\mu=1}^{N-1} |\mu)(\mu| + \sum_{\mu=1}^{N-1} T_\mu |\mu)(0|. \tag{37}$$

Proof. The proof of this proposition can be easily realized in matrix representation by using propositions 3 and 4. □

As a result we have that any linear real trace-preserving superoperator can be realized by three types of superoperators: (1) unital orthogonal superoperator $\hat{\mathcal{U}}$; (2) unital diagonal superoperator $\hat{D}$; (3) nonunital translation superoperator $\hat{\mathcal{E}}^{(T)}$.

Proposition 6. *If the quantum operation $\hat{\mathcal{E}}$ has the form*

$$\hat{\mathcal{E}}(\rho) = \sum_{j=1}^{m} A_j \rho A_j^\dagger$$

where A is a self-adjoint operator ($A_j^\dagger = A_j$), then quantum four-valued logic gate $\hat{\mathcal{E}}$ is described by symmetric matrix $\mathcal{E}_{\mu\nu} = \mathcal{E}_{\nu\mu}$. This gate is trace-preserving if $\mathcal{E}_{\mu 0} = \mathcal{E}_{0\mu} = \delta_{\mu 0}$.

Proof. If $A_j^\dagger = A_j$, then

$$\begin{aligned}\mathcal{E}_{\mu\nu} &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{m} \operatorname{Tr}(\sigma_\mu A_j \sigma_\nu A_j) \\ &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{m} \operatorname{Tr}(\sigma_\nu A_j \sigma_\mu A_j) = \mathcal{E}_{\nu\mu}.\end{aligned}$$

Using proposition 2 we have that this gate is trace-preserving if $\mathcal{E}_{\mu 0} = \mathcal{E}_{0\mu} = \delta_{\mu 0}$. □

5.4. Unitary two-valued logic gates as orthogonal four-valued logic gates

Let us consider a unitary two-valued logic gate defined by unitary operator U acting on pure states—unit elements of Hilbert space. The map $\hat{\mathcal{U}} : \rho \to U\rho U^\dagger$ induced by a unitary operator U is a particular case of quantum operation.

Proposition 7. *In the generalized computational basis any unitary quantum two-valued logic gate U acting on pure n-qubit states can be considered as a quantum four-valued logic gate $\hat{\mathcal{U}}$ acting on n-ququats:*

$$\hat{\mathcal{U}} = \sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1} \mathcal{U}_{\mu\nu} |\mu)(\nu| \tag{38}$$

where $\mathcal{U}_{\mu\nu}$ is a real matrix such that

$$\mathcal{U}_{\mu\nu} = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\nu U \sigma_\mu U^\dagger). \tag{39}$$

Proof. Using proposition 1 and the equation

$$|\rho') = \hat{\mathcal{U}}|\rho) = |U\rho U^\dagger)$$

we get this proposition.

Formulae (38) and (39) define a relation between the unitary quantum two-valued logic gate U and the real $4^n \times 4^n$ matrix $\mathcal{U}$ of quantum four-valued logic gate $\hat{\mathcal{U}}$. □

Proposition 8. *Any four-valued logic gate associated with a unitary two-valued logic gate by (38) and (39) is a unital gate, i.e. gate matrix $\mathcal{U}$ defined by (39) has $\mathcal{U}_{\mu 0} = \mathcal{U}_{0\mu} = \delta_{\mu 0}$.*

Proof.

$$\mathcal{U}_{\mu 0} = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\mu U \sigma_0 U^\dagger) = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\mu U U^\dagger) = \frac{1}{2^n}\,\mathrm{Tr}\,\sigma_\nu.$$

Using $\mathrm{Tr}\,\sigma_\mu = \delta_{\mu 0}$ we get $\mathcal{U}_{\mu 0} = \delta_{\mu 0}$.

□

Let us denote the gate $\hat{\mathcal{U}}$ associated with unitary two-valued logic gate U by $\hat{\mathcal{E}}^{(U)}$.

Proposition 9. *If U is a unitary two-valued logic gate, then in the generalized computational basis a quantum four-valued logic gate $\hat{\mathcal{U}} = \hat{\mathcal{E}}^{(U)}$ associated with U is represented by orthogonal matrix $\mathcal{E}^{(U)}$:*

$$\mathcal{E}^{(U)}\left(\mathcal{E}^{(U)}\right)^T = \left(\mathcal{E}^{(U)}\right)^T \mathcal{E}^{(U)} = I. \tag{40}$$

Proof. Let $\hat{\mathcal{E}}^{(U)}$ be defined by

$$\hat{\mathcal{E}}^{(U)}|\rho) = |U\rho U^\dagger) \qquad \hat{\mathcal{E}}^{(U^\dagger)}|\rho) = |U^\dagger \rho U).$$

If $UU^\dagger = U^\dagger U = I$, then

$$\hat{\mathcal{E}}^{(U)}\hat{\mathcal{E}}^{(U^\dagger)} = \hat{\mathcal{E}}^{(U^\dagger)}\hat{\mathcal{E}}^{(U)} = \hat{I}.$$

In the matrix representation we have

$$\sum_{\alpha=0}^{N-1} \mathcal{E}^{(U)}_{\mu\alpha}\mathcal{E}^{(U^\dagger)}_{\alpha\nu} = \sum_{\alpha=0}^{N-1} \mathcal{E}^{(U^\dagger)}_{\mu\alpha}\mathcal{E}^{(U)}_{\alpha\nu} = \delta_{\mu\nu}$$

i.e. $\mathcal{E}^{(U^\dagger)}\mathcal{E}^{(U)} = \mathcal{E}^{(U)}\mathcal{E}^{(U^\dagger)} = I$. Note that

$$\mathcal{E}^{(U^\dagger)}_{\mu\nu} = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\mu U^\dagger \sigma_\nu U) = \frac{1}{2^n}\,\mathrm{Tr}(\sigma_\nu U \sigma_\mu U^\dagger) = \mathcal{E}^{(U)}_{\nu\mu}$$

i.e. $\mathcal{E}^{(U^\dagger)} = (\mathcal{E}^{(U)})^T$. Finally, we obtain (40). □

Proposition 10. *If $\hat{\mathcal{E}}^\dagger$ is an adjoint superoperator for linear trace-preserving superoperator $\hat{\mathcal{E}}$, then the matrices of the superoperator are connected by transposition $\mathcal{E}^\dagger = \mathcal{E}^T$:*

$$(\mathcal{E}^\dagger)_{\mu\nu} = \mathcal{E}_{\nu\mu}. \tag{41}$$

Proof. Using

$$\hat{\mathcal{E}} = \sum_{j=1}^{m} \hat{L}_{A_j}\hat{R}_{A_j^\dagger} \qquad \hat{\mathcal{E}}^\dagger = \sum_{j=1}^{m} \hat{L}_{A_j^\dagger}\hat{R}_{A_j}$$

we get

$$\mathcal{E}_{\mu\nu} = \frac{1}{2^n}\sum_{j=1}^{m}\mathrm{Tr}(\sigma_\mu A_j \sigma_\nu A_j^\dagger)$$

$$(\mathcal{E}^\dagger)_{\mu\nu} = \frac{1}{2^n}\sum_{j=1}^{m}\mathrm{Tr}(\sigma_\mu A_j^\dagger \sigma_\nu A_j)$$

$$= \frac{1}{2^n}\sum_{j=1}^{m}\mathrm{Tr}(\sigma_\nu A_j \sigma_\mu A_j^\dagger) = \mathcal{E}_{\nu\mu}. \qquad \square$$

Obviously, if we define the superoperator $\hat{\mathcal{E}}$ by

$$\hat{\mathcal{E}} = \sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1}\mathcal{E}_{\mu\nu}|\mu)(\nu|$$

then the adjoint superoperator has the form

$$\hat{\mathcal{E}}^\dagger = \sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1}\mathcal{E}_{\nu\mu}|\mu)(\nu|.$$

Proposition 11. *If $\hat{\mathcal{E}}^\dagger\hat{\mathcal{E}} = \hat{\mathcal{E}}\hat{\mathcal{E}}^\dagger = \hat{I}$, then $\hat{\mathcal{E}}$ is an orthogonal quantum four-valued logic gate, i.e. $\mathcal{E}^T\mathcal{E} = \mathcal{E}\mathcal{E}^T = I$.*

Proof. If $\hat{\mathcal{E}}^\dagger\hat{\mathcal{E}} = \hat{I}$, then

$$\sum_{\alpha=0}^{N-1}(\mu|\hat{\mathcal{E}}^\dagger|\alpha)(\alpha|\hat{\mathcal{E}}|\nu) = (\mu|\hat{I}|\nu)$$

i.e.

$$\sum_{\alpha=0}^{N-1}(\mathcal{E}^\dagger)_{\mu\alpha}\mathcal{E}_{\alpha\nu} = \delta_{\mu\nu}.$$

Using proposition 8 we have

$$\sum_{\alpha=0}^{N-1}(\mathcal{E}^T)_{\mu\alpha}\mathcal{E}_{\alpha\nu} = \delta_{\mu\nu}$$

i.e. $\mathcal{E}^T\mathcal{E} = I$. $\square$

Note that n-qubit unitary two-valued logic gate U is an element of Lie group $SU(2^n)$. The dimension of this group is equal to $\dim SU(2^n) = (2^n)^2 - 1 = 4^n - 1$. The matrix of n-ququat orthogonal linear gate $\hat{\mathcal{U}} = \hat{\mathcal{E}}^{(U)}$ can be considered as an element of Lie group $SO(4^n - 1)$. The dimension of this group is equal to $\dim SO(4^n - 1) = (4^n - 1)(2 \times 4^{n-1} - 1)$.

For example, if $n = 1$, then

$$\dim SU(2^1) = 3 \qquad \dim SO(4^1 - 1) = 3.$$

If $n = 2$, then

$$\dim SU(2^2) = 15 \qquad \dim SO(4^2 - 1) = 105.$$

Therefore, not all orthogonal quantum four-valued logic gates for mixed and pure states are connected with unitary two-valued logic gates for pure states.

5.5. Single ququat orthogonal gates

Let us consider single ququat quantum four-valued logic gate $\hat{\mathcal{U}}$ associated with unitary single qubit two-valued logic gate U.

Proposition 12. *Any single qubit unitary quantum two-valued logic gate can be realized as the product of single ququat simple rotation quantum four-valued logic gates* $\hat{\mathcal{U}}^{(1)}(\alpha)$, $\hat{\mathcal{U}}^{(2)}(\theta)$ *and* $\hat{\mathcal{U}}^{(1)}(\beta)$ *defined by*

$$\hat{\mathcal{U}}^{(1)}(\alpha) = |0)(0| + |3)(3| + \cos\alpha(|1)(1| + |2)(2|) + \sin\alpha(|2)(1| - |1)(2|)$$

$$\hat{\mathcal{U}}^{(2)}(\theta) = |0)(0| + |2)(2| + \cos\theta(|1)(1| + |3)(3|) + \sin\theta(|1)(3| - |3)(1|)$$

where α, θ *and* β *are Euler angles.*

Proof. Let us consider a general single qubit unitary gate [56]. Every unitary one-qubit gate U can be represented by a 2×2-matrix

$$U(\alpha, \theta, \beta) = \mathrm{e}^{-\mathrm{i}\alpha\sigma_3/2}\,\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2}\,\mathrm{e}^{-\mathrm{i}\beta\sigma_3/2} = U_1(\alpha)U_2(\theta)U_1(\beta)$$

where

$$U_1(\alpha) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha/2} \end{pmatrix} \qquad U_2(\theta) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

where α, θ and β are Euler angles. The corresponding 4×4-matrix $\mathcal{U}(\alpha, \theta, \beta)$ of a four-valued logic gate has the form

$$\mathcal{U}(\alpha, \theta, \beta) = \mathcal{U}^{(1)}(\alpha)\mathcal{U}^{(2)}(\theta)\mathcal{U}^{(1)}(\beta)$$

where

$$\mathcal{U}^{(1)}_{\mu\nu}(\alpha) = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu U_1(\alpha)\sigma_\nu U_1^\dagger(\alpha))$$

$$\mathcal{U}^{(2)}_{\mu\nu}(\theta) = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu U_2(\theta)\sigma_\nu U_2^\dagger(\theta)).$$

Finally, we obtain

$$\mathcal{U}^{(1)}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{U}^{(2)}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

where

$$0 \leqslant \alpha < 2\pi \qquad 0 \leqslant \theta \leqslant \pi \qquad 0 \leqslant \beta \leqslant 2\pi.$$

Using $U(\alpha, \theta + 2\pi, \beta) = -U(\alpha, \theta, \beta)$, we get that two-valued logic gates $U(\alpha, \theta, \beta)$ and $U(\alpha, \theta + 2\pi, \beta)$ map to single quantum four-valued logic gate $\mathcal{U}(\alpha, \theta, \beta)$. The back rotation four-valued logic gate is defined by the matrix

$$\mathcal{U}^{-1}(\alpha, \theta, \beta) = \mathcal{U}(2\pi - \alpha, \pi - \theta, 2\pi - \beta).$$

The simple rotation gates $\hat{\mathcal{U}}^{(1)}(\alpha)$, $\hat{\mathcal{U}}^{(2)}(\theta)$, $\hat{\mathcal{U}}^{(1)}(\beta)$ are defined by matrices $\hat{\mathcal{U}}^{(1)}(\alpha)$, $\hat{\mathcal{U}}^{(2)}(\theta)$ and $\hat{\mathcal{U}}^{(1)}(\beta)$. □

Let us introduce simple reflection gates by

$$\hat{\mathcal{R}}^{(1)} = |0)(0| - |1)(1| + |2)(2| + |3)(3|$$

$$\hat{\mathcal{R}}^{(2)} = |0)(0| + |1)(1| - |2)(2| + |3)(3|$$

$$\hat{\mathcal{R}}^{(3)} = |0)(0| + |1)(1| + |2)(2| - |3)(3|.$$

Proposition 13. *Any single ququat linear quantum four-valued logic gate $\hat{\mathcal{E}}$ defined by orthogonal matrix $\mathcal{E} : \mathcal{E}\mathcal{E}^T = I$ can be realized by*

- *simple rotation gates $\hat{\mathcal{U}}^{(1)}$ and $\hat{\mathcal{U}}^{(2)}$.*
- *inversion gate $\hat{\mathcal{I}}$ defined by*

$$\hat{\mathcal{I}} = |0)(0| - |1)(1| - |2)(2| - |3)(3|.$$

Proof. Using proposition 10 and

$$\hat{\mathcal{R}}^{(3)} = \hat{\mathcal{U}}^{(1)}\hat{\mathcal{I}} \qquad \hat{\mathcal{R}}^{(2)} = \hat{\mathcal{U}}^{(2)}\hat{\mathcal{I}} \qquad \hat{\mathcal{R}}^{(1)} = \hat{\mathcal{U}}^{(1)}\hat{\mathcal{U}}^{(1)}\hat{\mathcal{I}}$$

we get this proposition. □

Example 1. In the generalized computational basis the Pauli matrices as two-valued logic gates are the quantum four-valued logic gates with diagonal 4×4 matrices. The gate $I = \sigma_0$ is

$$\hat{\mathcal{U}}^{(\sigma 0)} = \sum_{\mu=0}^{3} |\mu)(\mu| = \hat{I}$$

i.e. $\mathcal{U}^{(\sigma 0)}_{\mu\nu} = \frac{1}{2}\operatorname{Tr}(\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}$.

For the unitary quantum two-valued logic gates which are equal to the Pauli matrix σ_k, where $k \in \{1, 2, 3\}$, we have quantum four-valued logic gates

$$\hat{\mathcal{U}}^{(\sigma k)} = \sum_{\mu,\nu=0}^{3} \mathcal{U}^{(\sigma k)}_{\mu\nu} |\mu)(\nu|$$

with the matrix

$$\mathcal{U}^{(\sigma k)}_{\mu\nu} = 2\delta_{\mu 0}\delta_{\nu 0} + 2\delta_{\mu k}\delta_{\nu k} - \delta_{\mu\nu}. \tag{42}$$

Example 2. In the generalized computational basis the unitary NOT gate ('negation') of two-valued logic

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is represented by the quantum four-valued logic gate

$$\hat{\mathcal{U}}^{(X)} = |0)(0| + |1)(1| - |2)(2| - |3)(3|.$$

Example 3. The Hadamar two-valued logic gate

$$H = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$$

can be represented as a quantum four-valued logic gate by

$$\hat{\mathcal{E}}^{(H)} = |0)(0| - |2)(2| + |3)(1| + |1)(3|$$

with the matrix

$$\mathcal{E}^{(H)}_{\mu\nu} = \delta_{\mu 0}\delta_{\nu 0} - \delta_{\mu 2}\delta_{\nu 2} + \delta_{\mu 3}\delta_{\nu 1} + \delta_{\mu 1}\delta_{\nu 3}.$$

5.6. Measurements as quantum four-valued logic gates

It is known that the von Neumann measurement superoperator $\hat{\mathcal{E}}$ is defined by

$$\hat{\mathcal{E}}|\rho) = \sum_{k=1}^{r} |P_k \rho P_k) \tag{43}$$

where $\{P_k | k = 1, \ldots, r\}$ is a (not necessarily complete) sequence of orthogonal projection operators on $\mathcal{H}^{(n)}$.

Let P_k be projectors onto the pure state $|k\rangle$ which define the usual computational basis $\{|k\rangle\}$, i.e.

$$P_k = |k\rangle\langle k|.$$

Proposition 14. *A nonlinear quantum four-valued logic gate $\hat{\mathcal{N}}$ for von Neumann measurement (43) of the state $\rho = \sum_{\alpha=0}^{N-1} |\alpha)\rho_\alpha$ is defined by*

$$\hat{\mathcal{N}} = \sum_{k=1}^{r} \frac{1}{p(r)} \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} \mathcal{E}^{(k)}_{\mu\nu} |\mu)(\nu| \tag{44}$$

where

$$\mathcal{E}^{(k)}_{\mu\nu} = \frac{1}{2^n} \mathrm{Tr}(\sigma_\mu P_k \sigma_\nu P_k) \qquad p(r) = \sqrt{2^n} \sum_{k=1}^{r} \sum_{\alpha=0}^{N-1} \mathcal{E}^{(k)}_{0\alpha} \rho_\alpha. \tag{45}$$

Proof. The trace-decreasing superoperator $\hat{\mathcal{E}}_k$ is defined by

$$|\rho) \to |\rho') = \hat{\mathcal{E}}_k|\rho) = |P_k \rho P_k).$$

This superoperator has the form $\hat{\mathcal{E}}_k = \hat{L}_{P_k} \hat{R}_{P_k}$. Then

$$\rho'_\mu = (\mu|\rho') = (\mu|\hat{\mathcal{E}}_k|\rho) = \sum_{\nu=0}^{N-1} (\mu|\hat{\mathcal{E}}_k|\nu)(\nu|\rho) = \sum_{\nu=0}^{N-1} \mathcal{E}^{(k)}_{\mu\nu} \rho_\nu$$

where

$$\mathcal{E}^{(k)}_{\mu\nu} = (\mu|\hat{\mathcal{E}}_k|\nu) = \frac{1}{2^n} \mathrm{Tr}(\sigma_\mu P_k \sigma_\nu P_k).$$

The probability that the process represented by $\hat{\mathcal{E}}_k$ occurs is

$$p(k) = \mathrm{Tr}(\hat{\mathcal{E}}_k(\rho)) = (I|\hat{\mathcal{E}}_k|\rho) = \sqrt{2^n} \rho'_0 = \sqrt{2^n} \sum_{\alpha=0}^{N-1} \mathcal{E}^{(k)}_{0\alpha} \rho_\alpha.$$

If

$$p(r) = \sqrt{2^n} \sum_{k=1}^{r} \sum_{\alpha=0}^{N-1} \mathcal{E}^{(k)}_{0\alpha} \rho_\alpha \neq 0$$

then the matrix for nonlinear trace-preserving gate $\hat{\mathcal{N}}$ is

$$\mathcal{N}_{\mu\nu} = \sum_{k=1}^{r} p^{-1}(r) \mathcal{E}^{(k)}_{\mu\nu}.$$

□

Example. Let us consider the single ququat projection operator

$$P_0 = |0\rangle\langle 0| = \tfrac{1}{2}(\sigma_0 + \sigma_3).$$

Table 1. Single argument classical gates.

x	$\sim x$	$\Box x$	$\Diamond x$	$\overline{x}$	I_0	I_1	I_2	I_3
0	3	0	0	1	3	0	0	0
1	2	0	3	2	0	3	0	0
2	1	0	3	3	0	0	3	0
3	0	3	3	0	0	0	0	3

Using formula (45) we derive

$$\mathcal{E}^{(0)}_{\mu\nu} = \frac{1}{8}\,\mathrm{Tr}(\sigma_\mu(\sigma_0+\sigma_3)\sigma_\nu(\sigma_0+\sigma_3))$$
$$= \frac{1}{2}(\delta_{\mu 0}\delta_{\nu 0} + \delta_{\mu 3}\delta_{\nu 3} + \delta_{\mu 3}\delta_{\nu 0} + \delta_{\mu 0}\delta_{\nu 3}).$$

The linear trace-decreasing superoperator for von Neumann measurement projector $|0\rangle\langle 0|$ onto the pure state $|0\rangle$ is

$$\hat{\mathcal{E}}^{(0)} = \frac{1}{2}(|0)(0| + |3)(3| + |0)(3| + |3)(0|).$$

Example. For the projection operator

$$P_1 = |1\rangle\langle 1| = \frac{1}{2}(\sigma_0 - \sigma_3).$$

Using formula (45) we derive

$$\mathcal{E}^{(1)}_{\mu\nu} = \frac{1}{2}(\delta_{\mu 0}\delta_{\nu 0} + \delta_{\mu 3}\delta_{\nu 3} - \delta_{\mu 3}\delta_{\nu 0} - \delta_{\mu 0}\delta_{\nu 3}).$$

The linear superoperator $\hat{\mathcal{E}}^{(1)}$ for the von Neumann measurement projector onto the pure state $|1\rangle$ is

$$\hat{\mathcal{E}}^{(1)} = \frac{1}{2}(|0)(0| + |3)(3| - |0)(3| - |3)(0|).$$

The superoperators $\hat{\mathcal{E}}^{(0)}$ and $\hat{\mathcal{E}}^{(1)}$ are not trace-preserving. The probabilities that processes represented by superoperators $\hat{\mathcal{E}}^{(k)}$ occur are

$$p(0) = \frac{1}{\sqrt{2}}(\rho_0 + \rho_3) \qquad p(1) = \frac{1}{\sqrt{2}}(\rho_0 - \rho_3).$$

6. Classical four-valued logic gates

Let us consider some elements of classical four-valued logic. For the concept of many-valued logic, see [49–53].

6.1. Elementary classical gates

A function $g(x_1, \ldots, x_n)$ describes a classical four-valued logic gate if the following conditions hold:

- all $x_i \in \{0, 1, 2, 3\}$, where $i = 1, \ldots, n$.
- $g(x_1, \ldots, x_n) \in \{0, 1, 2, 3\}$.

It is known that the number of all classical logic gates with n arguments $x_1, \ldots, x_n$ is equal to 4^{4^n}. The number of classical logic gates $g(x)$ with single argument is equal to $4^{4^1} = 256$. Let us write some of these gates in table 1.

The number of classical logic gates $g(x_1, x_2)$ with two arguments is equal to

$$4^{4^2} = 4^{16} = 42\,949\,677\,296.$$

Let us write some of these classical gates in table 2.

Table 2. Two-argument classical gates.

(x_1, x_2)	$\wedge$	$\vee$	V_4	$\sim V_4$
(0, 0)	0	0	1	2
(0, 1)	0	1	2	1
(0, 2)	0	2	3	0
(0, 3)	0	3	0	3
(1, 0)	0	1	2	1
(1, 1)	1	1	2	1
(1, 2)	1	2	3	0
(1, 3)	1	3	0	3
(2, 0)	0	2	3	0
(2, 1)	1	2	3	0
(2, 2)	2	2	3	0
(2, 3)	2	3	0	3
(3, 0)	0	3	0	3
(3, 1)	1	3	0	3
(3, 2)	2	3	0	3
(3, 3)	3	3	0	3

Let us define some elementary classical four-valued logic gates by formulae:

- Luckasiewicz negation: $\sim x = 3 - x$.
- Cyclic shift: $\overline{x} = x + 1 (\mathrm{mod}\, 4)$.
- Functions $I_i(x)$, where $i = 0, \ldots, 3$, such that $I_i(x) = 3$ if $x = i$ and $I_i(x) = 0$ if $x \neq i$.
- Generalized conjunction: $x_1 \wedge x_2 = \min(x_1, x_2)$.
- Generalized disjunction: $x_1 \vee x_2 = \max(x_1, x_2)$.
- Generalized Sheffer–Webb function:

$$V_4(x_1, x_2) = \max(x_1, x_2) + 1 (\mathrm{mod}\, 4).$$

The generalized conjunction and disjunction satisfy the commutative law, associative law and distributive law:

- Commutative law

$$x_1 \wedge x_2 = x_2 \wedge x_1 \qquad x_1 \vee x_2 = x_2 \vee x_1.$$

- Associative law

$$(x_1 \vee x_2) \vee x_3 = x_1 \vee (x_2 \vee x_3)$$
$$(x_1 \wedge x_2) \wedge x_3 = x_1 \wedge (x_2 \wedge x_3).$$

- Distributive law

$$x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3)$$
$$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3).$$

Note that the Luckasiewicz negation satisfies the following properties:

$$\sim(\sim x) = x \qquad \sim(x_1 \wedge x_2) = (\sim x_1) \vee (\sim x_2).$$

The following usual negation rules are not satisfied by the circular shift:

$$\overline{\overline{x}} \neq x \qquad \overline{x_1 \wedge x_2} \neq \overline{x_1} \vee \overline{x_2}.$$

The analogue of the disjunction normal form of the n-argument classical four-valued logic gate is

$$g(x_1, \ldots, x_n) = \bigvee_{(k_1,\ldots,k_n)} I_{k_1}(x_1) \wedge \cdots \wedge I_{k_n}(x_n) \wedge g(k_1, \ldots, k_n).$$

Let us consider (functional) complete sets [51, 52] of classical four-valued logic gates.

Proposition 15. *The set* $\{0, 1, 2, 3, I_0, I_1, I_2, I_3, x_1 \wedge x_2, x_1 \vee x_2\}$ *is a complete set.*
The set $\{\overline{x}, x_1 \vee x_2\}$ *is a complete set.*
The gate $V_4(x_1, x_2)$ *is a complete set.*

Proof. This proposition is proved in [52]. □

6.2. Quantum gates for single argument classical gates

Let us consider linear trace-preserving quantum gates for classical gates $\sim$, $\overline{x}$, I_0, I_1, I_2, I_3, 0, 1, 2, 3, $\Diamond$, $\Box$.

Proposition 16. *Any single argument classical four-valued logic gate* $g(\nu)$ *can be realized as a linear trace-preserving quantum four-valued logic gate by*

$$\hat{\mathcal{E}}^{(g)} = |0)(0| + \sum_{k=1}^{3} |g(k))(k| + \left(1 - \delta_{0g(0)}\right) \left(|g(0))(0| - \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \left(1 - \delta_{\mu g(\nu)}\right) |\mu)(\nu| \right). \quad (46)$$

Proof. The proof is by direct calculation in

$$\hat{\mathcal{E}}^{(g)}|\alpha] = |g(\alpha)]$$

where

$$\hat{\mathcal{E}}^{(g)}|\alpha] = \frac{1}{\sqrt{2}} \left(\hat{\mathcal{E}}^{(g)}|0) + \hat{\mathcal{E}}^{(g)}|\alpha)\right).$$

□

Examples.

1. The Luckasiewicz negation gate is

$$\hat{\mathcal{E}}^{(\sim)} = |0)(0| + |1)(2| + |2)(1| + |3)(0| - |3)(3|.$$

2. The four-valued logic gate I_0 can be realized by

$$\hat{\mathcal{E}}^{(I_0)} = |0)(0| + |3)(0| - \sum_{k=1}^{3} |3)(k|.$$

3. The gates $I_k(x)$, where $k = 1, 2, 3$, are

$$\hat{\mathcal{E}}^{(I_k)} = |0)(0| + |3)(k|.$$

4. The gate $\overline{x}$ can be realized by

$$\hat{\mathcal{E}}^{(\overline{x})} = |0)(0| + |1)(0| + |2)(1| + |3)(2| - \sum_{k=1}^{3} |1)(k|.$$

5. The constant gates 0 and $k = 1, 2, 3$ can be realized by

$$\hat{\mathcal{E}}^{(0)} = |0)(0| \qquad \hat{\mathcal{E}}^{(k)} = |0)(0| + |k)(0|.$$

6. The gate $\Diamond x$ is realized by

$$\hat{\mathcal{E}}^{(\Diamond)} = |0)(0| + \sum_{k=1}^{3} |3)(k|.$$

7. The gate $\Box x = \sim \Diamond x$ is

$$\hat{\mathcal{E}}^{(\Box)} = |0)(0| + |3)(3|.$$

Note that the quantum four-valued logic gates $\hat{\mathcal{E}}^{(\sim)}$, $\hat{\mathcal{E}}^{(I_0)}$, $\hat{\mathcal{E}}^{(k)}$, $\hat{\mathcal{E}}^{(g_1)}$ are not unital gates.

6.3. Quantum gates for two-argument classical gates

Let us consider quantum four-valued logic gates for two-argument classical four-valued logic gates.

1. The generalized conjunction $x_1 \wedge x_2 = \min(x_1, x_2)$ and generalized disjunction $x_1 \wedge x_2 = \max(x_1, x_2)$ can be realized by a two-ququat quantum four-valued logic gate with $T = 0$:

$$\hat{\mathcal{E}}|x_1, x_2] = |x_1 \vee x_2, x_1 \wedge x_2].$$

Let us write the quantum four-valued logic gate which realizes the gate in the generalized computational basis by

$$\hat{\mathcal{E}} = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} |\mu\nu)(\mu\nu| + \sum_{k=1}^{3} (|0k) - |k0))(k0| + \sum_{k=2}^{3} (|1k) - |k1))(k1| + (|23) - |32))(32|.$$

2. The Sheffer–Webb function gate $|x_1, x_2] \to |V_4(x_1, x_2), \sim V_4(x_1, x_2)]$ can be realized by a two-ququat quantum gate with $T \neq 0$:

$$\hat{\mathcal{E}}^{(SW)} = |00)(00| + |12)(00| - \sum_{\mu=0}^{3} \sum_{\nu=1}^{3} |12)(\mu\nu| + |21)(10| + |21)(11| + |30)(02| + |30)(20| + |30)(12| + |30)(21| + |30)(22| + |03)(03| + |03)(13| + |03)(23| + \sum_{\mu=0}^{3} |03)(3\mu|.$$

Note that this Sheffer–Webb function gate is not a unital quantum gate and

$$\hat{\mathcal{E}}^{(SW)} \neq |V_4(x_1, x_2), \sim V_4(x_1, x_2))(x_1, x_2|.$$

7. Universal set of quantum four-valued logic gates

The condition for performing arbitrary unitary operations to realize a quantum computation by dynamics of a closed quantum system is well understood [56–59]. Using quantum unitary gates, a quantum computer with pure states may realize the time sequence of operations corresponding to any unitary dynamics. Deutsch *et al* [57], DiVincenzo [58] and Lloyd [59] showed that almost any two-qubit quantum unitary gate is universal for a quantum computer with pure states. It is known [56–59] that a set of quantum gates that consists of all one-qubit unitary gates and the two-qubit exclusive-OR (XOR) gate is universal for a quantum computer with pure states in the sense that all unitary operations on arbitrarily many qubits can be expressed as compositions of these unitary gates. Recently in [43] universality for a quantum computer with n-qubit quantum unitary gates on pure states was considered.

The same is not true for the general quantum operations (superoperators) corresponding to the dynamics of open quantum systems. In the paper [20] a single qubit open quantum system with Markovian dynamics was considered and the resources needed for universality of general quantum operations were studied. An analysis of completely positive trace-preserving superoperators on single qubit density matrices was realized in papers [66–68].

Let us study universality for quantum four-valued logic gates [2, 3].

Definition. *A set of quantum four-valued logic gates is universal iff all quantum gates on arbitrarily many ququats can be expressed as compositions of these gates.*

Single ququat gates cannot map two initially un-entangled ququats into an entangled state. Therefore, the single ququat gates or set of single ququat gates are not universal gates for a quantum computer with mixed states. Quantum gates which are realizations of classical gates cannot be universal by definition, since these gates evolve generalized computational states to generalized computational states and never to the superposition of them.

The matrix $\mathcal{E}$ of the linear real superoperator $\hat{\mathcal{E}}$ on $\overline{\mathcal{H}}^{(n)}$ is an element of Lie group $TGL(4^n-1,\mathbb{R})$. The linear superoperator $\hat{\mathcal{E}}$ on $\overline{\mathcal{H}}^{(n)}$ is a quantum four-valued logic gate (completely positive trace-preserving superoperator) iff the matrix $\mathcal{E}$ is a completely positive element of Lie group $TGL(4^n-1,\mathbb{R})$. The matrix $\mathcal{N}$ of a nonlinear real trace-preserving superoperator $\hat{\mathcal{N}}$ on $\overline{\mathcal{H}}^{(n)}$ is a quantum four-valued logic gate defined by

$$\hat{\mathcal{N}}(\rho)=\frac{\hat{\mathcal{E}}(\rho)}{\mathrm{Tr}(\hat{\mathcal{E}}(\rho))} \tag{47}$$

iff the matrix $\mathcal{E}$ of the linear trace-decreasing superoperator $\hat{\mathcal{E}}$ is a completely positive element of Lie group $GL(4^n,\mathbb{R})$. The condition of complete positivity leads to difficult inequalities for matrix elements [65–68]. In order to satisfy the condition of complete positivity we use the following representation:

$$\hat{\mathcal{E}}=\sum_{j=1}^{m}\hat{L}_{A_j}\hat{R}_{A_j^\dagger} \tag{48}$$

where $\hat{L}_A$ and $\hat{R}_A$ are left and right multiplication superoperators on $\overline{\mathcal{H}}^{(n)}$ defined by $\hat{L}_A|B)=|AB)$, $\hat{R}_A|B)=|BA)$. It is known that any linear completely positive superoperator $\hat{\mathcal{E}}$ can be represented by (48). Any trace-decreasing superoperator (48) generates a quantum four-valued logic gate by (47). To find the universal set of completely positive (linear or nonlinear) superoperators, i.e. quantum four-valued logic gates, we suggest considering the universal set of the superoperators $\hat{L}_{A_j}$ and $\hat{R}_{A_j^\dagger}$. Let the superoperators $\hat{L}_{A_j}$ and $\hat{R}_{A_j^\dagger}$ be called pseudo-gates. A set of pseudo-gates is universal iff all pseudo-gates on arbitrarily many ququats can be expressed as compositions of these pseudo-gates. The matrices of the superoperators $\hat{L}_A$ and $\hat{R}_{A^\dagger}$ are connected by complex conjugation. The set of these matrices is a group $GL(4^n,\mathbb{C})$. Obviously, the universal set of pseudo-gates $\hat{L}_A$ is connected with a universal set of completely positive superoperators $\hat{\mathcal{E}}$ of the quantum four-valued logic gates.

The trace-preserving condition for linear superoperator (48) is equivalent to the requirement $\mathcal{E}_{0\mu}=\delta_{0\mu}$ for gate matrix $\mathcal{E}$. The trace-decreasing condition can be satisfied by inequality of the following proposition.

Proposition 17. *If the matrix elements $\mathcal{E}_{\mu\nu}$ of a superoperator $\hat{\mathcal{E}}$ are satisfied by the inequality*

$$\sum_{\mu=0}^{N-1}(\mathcal{E}_{0\mu})^2\leqslant 1 \tag{49}$$

then $\hat{\mathcal{E}}$ is a trace-decreasing superoperator.

Proof. Using the Schwarz inequality

$$\left(\sum_{\mu=0}^{N-1} \mathcal{E}_{0\mu}\rho_\mu\right)^2 \leqslant \sum_{\mu=0}^{N-1}(\mathcal{E}_{0\mu})^2 \sum_{\nu=0}^{N-1}(\rho_\nu)^2$$

and the property of the density matrix

$$\operatorname{Tr}\rho^2 = (\rho|\rho) = \sum_{\nu=0}^{N-1}(\rho_\nu)^2 \leqslant 1$$

we have

$$|\operatorname{Tr}\hat{\mathcal{E}}(\rho)|^2 = |(0|\hat{\mathcal{E}}|\rho)|^2 = \left(\sum_{\mu=0}^{N-1} \mathcal{E}_{0\mu}\rho_\mu\right)^2 \leqslant \sum_{\mu=0}^{N-1}(\mathcal{E}_{0\mu})^2.$$

Using (49), we get $|\operatorname{Tr}\hat{\mathcal{E}}(\rho)| \leqslant 1$. Since $\hat{\mathcal{E}}$ is a completely positive (or positive) superoperator ($\hat{\mathcal{E}}(\rho) \geqslant 0$), it follows that

$$0 \leqslant \operatorname{Tr}\hat{\mathcal{E}}(\rho) \leqslant 1$$

i.e. $\hat{\mathcal{E}}$ is a trace-decreasing superoperator. □

Let us consider the superoperators $\hat{L}_A$ and $\hat{R}_{A^\dagger}$. These superoperators can be represented by

$$\hat{L}_A = \sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1} L^{(A)}_{\mu\nu}|\mu)(\nu| \qquad \hat{R}_{A^\dagger} = \sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1} R^{(A^\dagger)}_{\mu\nu}|\mu)(\nu| \tag{50}$$

where matrices $L^{(A)}_{\mu\nu}$ and $R^{(A^\dagger)}_{\mu\nu}$ are defined by

$$L^{(A)}_{\mu\nu} = \frac{1}{2^n}\operatorname{Tr}(\sigma_\mu A\sigma_\nu) = \frac{1}{2^n}\operatorname{Tr}(\sigma_\alpha\sigma_\mu A)$$

$$R^{(A^\dagger)}_{\mu\nu} = \frac{1}{2^n}\operatorname{Tr}(\sigma_\mu\sigma_\nu A^\dagger) = \frac{1}{2^n}\operatorname{Tr}(A^\dagger\sigma_\mu\sigma_\nu).$$

Proposition 18. *The matrix $\mathcal{E}_{\mu\nu}$ of the completely positive superoperator (48) can be represented by*

$$\mathcal{E}_{\mu\nu} = \sum_{j=1}^{m}\sum_{\alpha=0}^{N-1} L^{(jA)}_{\mu\alpha} R^{(jA^\dagger)}_{\alpha\nu}. \tag{51}$$

Proof. Let us write the matrix $\mathcal{E}_{\mu\nu}$ by matrices of superoperators $\hat{L}_{A_j}$ and $\hat{R}_{A_j}$

$$\mathcal{E}_{\mu\nu} = (\mu|\hat{\mathcal{E}}|\nu) = \sum_{j=1}^{m}\big(\mu\big|\hat{L}_{A_j}\hat{R}_{A_j^\dagger}\big|\nu\big) = \sum_{j=1}^{m}\sum_{\alpha=0}^{N-1}\big(\mu\big|\hat{L}_{A_j}\big|\alpha\big)\big(\alpha\big|\hat{R}_{A_j^\dagger}\big|\nu\big) = \sum_{j=1}^{m}\sum_{\alpha=0}^{N-1} L^{(jA)}_{\mu\alpha}R^{(jA^\dagger)}_{\alpha\nu}.$$

Finally, we obtain (51), where

$$L^{(jA)}_{\mu\alpha} = (\mu|\hat{L}_A|\alpha) = \frac{1}{2^n}\big(\sigma_\mu\big|\hat{L}_{A_j}\big|\sigma_\alpha\big) = \frac{1}{2^n}\operatorname{Tr}(\sigma_\mu A_j\sigma_\alpha) = \frac{1}{2^n}\operatorname{Tr}(\sigma_\alpha\sigma_\mu A_j)$$

$$R^{(jA^\dagger)}_{\alpha\nu} = \big(\alpha\big|\hat{R}_{A_j^\dagger}\big|\nu\big) = \frac{1}{2^n}\big(\sigma_\alpha\big|\hat{R}_{A_j^\dagger}\big|\sigma_\nu\big) = \frac{1}{2^n}\operatorname{Tr}(\sigma_\alpha\sigma_\nu A_j^\dagger) = \frac{1}{2^n}\operatorname{Tr}(A_j^\dagger\sigma_\alpha\sigma_\nu).$$

The matrix elements can be rewritten in the form

$$L^{(jA)}_{\mu\alpha} = \frac{1}{2^n}(\sigma_\mu\sigma_\alpha|A_j) \qquad R^{(jA^\dagger)}_{\alpha\nu} = \frac{1}{2^n}(A_j|\sigma_\alpha\sigma_\nu). \tag{52}$$

□

Example. Let us consider the single ququat pseudo-gate $\hat{L}_A$. The elements of pseudo-gate matrix $L^{(A)}$ are defined by

$$L^{(A)}_{\mu\nu} = \tfrac{1}{2}\operatorname{Tr}(\sigma_\mu A\sigma_\nu).$$

Let us denote

$$a_\mu = \tfrac{1}{2}\operatorname{Tr}(\sigma_\mu A).$$

Using

$$L^{(A)}_{kl} = \frac{1}{2}\operatorname{Tr}(\sigma_l\sigma_k A) = \frac{1}{2}\delta_{kl}\operatorname{Tr} A + \frac{\mathrm{i}}{2}\varepsilon_{lkm}\operatorname{Tr}(\sigma_m A)$$

where $k, l, m = 1, 2, 3$, we get

$$\hat{L}_A = \sum_{\mu=0}^{3} a_0|\mu)(\mu| + \sum_{k=0}^{3} a_k(|0)(k| + |k)(0|) + \mathrm{i}a_1(|3)(2| - |2)(3|)$$
$$+ \mathrm{i}a_2(|1)(3| - |3)(1|) + \mathrm{i}a_3(|2)(1| - |1)(2|).$$

The pseudo-gate matrix is

$$L^{(A)}_{\mu\nu} = \delta_{\mu\nu}\operatorname{Tr} A + \sum_{m=1}^{3}(\delta_{\mu 0}\delta_{\nu m} + \delta_{\mu m}\delta_{\nu 0})\operatorname{Tr}(\sigma_m A) + \mathrm{i}\sum_{m=1}^{3}\delta_{\mu k}\delta_{\nu l}\varepsilon_{lkm}\operatorname{Tr}(\sigma_m A). \tag{53}$$

Let us consider the properties of the matrix elements $L^{(jA)}_{\mu\alpha}$ and $R^{(jA^\dagger)}_{\mu\alpha}$.

Proposition 19. *The matrices* $L^{(jA)}_{\mu\alpha}$ *and* $R^{(jA^\dagger)}_{\mu\alpha}$ *are complex* $4^n \times 4^n$ *matrices and their elements are connected by complex conjugation:*

$$\left(L^{(jA)}_{\mu\alpha}\right)^* = R^{(jA^\dagger)}_{\mu\alpha}. \tag{54}$$

Proof. Using complex conjugation of the matrix elements (52), we get

$$\left(L^{(jA)}_{\mu\alpha}\right)^* = \frac{1}{2^n}(\sigma_\mu\sigma_\alpha|A_j)^* = \frac{1}{2^n}(A_j|\sigma_\mu\sigma_\alpha) = R^{(jA^\dagger)}_{\mu\alpha}.$$ □

We can write the gate matrix (51) in the form

$$\mathcal{E}_{\mu\nu} = \sum_{j=1}^{m}\sum_{\alpha=0}^{N-1} L^{(jA)}_{\mu\alpha}\left(L^{(jA)}_{\alpha\nu}\right)^*. \tag{55}$$

Proposition 20. *The matrices* $L^{(jA)}_{\mu\alpha}$ *and* $R^{(jA^\dagger)}_{\mu\alpha}$ *of the n-ququat quantum four-valued logic gate (48) are the elements of Lie group* $GL(4^n, \mathbb{C})$*. The set of these matrices is a group.*

Proof. The proof is trivial. □

A superoperator $\hat{\mathcal{E}}$ on $\overline{\mathcal{H}}^{(2)}$ is called primitive [43] if $\hat{\mathcal{E}}$ maps the tensor product of single ququats to the tensor product of single ququats, i.e. if $|\rho_1)$ and $|\rho_2)$ are ququats, then we can find ququats $|\rho_1')$ and $|\rho_2')$ such that

$$\hat{\mathcal{E}}|\rho_1 \otimes \rho_2) = |\rho_1' \otimes \rho_2').$$

The superoperator $\hat{\mathcal{E}}$ is called imprimitive if $\hat{\mathcal{E}}$ is not primitive.

It can be shown that almost every pseudo-gate that operates on two or more ququats is a universal pseudo-gate.

Proposition 21. *The set of all single ququat pseudo-gates and any imprimitive two-ququat pseudo-gate are universal sets of pseudo-gates.*

Proof. This proposition can be proved by analogy with [43, 57, 58]. Let us consider some points of the proof. Expressed in group theory language, all n-ququat pseudo-gates are elements of the Lie group $GL(4^n, \mathbb{C})$. Two-ququat pseudo-gates $\hat{L}$ are elements of Lie group $GL(16, \mathbb{C})$. The question of universality is the same as that of what set of superoperators $\hat{L}$ is sufficient to generate $GL(16, \mathbb{C})$. The group $GL(16, \mathbb{C})$ has $(16)^2 = 256$ independent one-parameter subgroups $GL_{\mu\nu}(16, \mathbb{C})$ of one-parameter pseudo-gates $\hat{L}^{(\mu\nu)}(t)$ such that $\hat{L}^{(\mu\nu)}(t) = t|\mu)(\nu|$. Infinitesimal generators of Lie group $GL(4^n, \mathbb{C})$ are defined by

$$\hat{H}_{\mu\nu} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}^{(\mu\nu)}(t)\right)_{t=0} \tag{56}$$

where $\mu, \nu = 0, 1, \ldots, 4^n - 1$. The generators $\hat{H}_{\mu\nu}$ of the one-parameter subgroup $GL_{\mu\nu}(4^n, \mathbb{R})$ are superoperators of the form $\hat{H}_{\mu\nu} = |\mu)(\nu|$ on $\overline{\mathcal{H}}^{(n)}$ which can be represented by $4^n \times 4^n$ matrices $H_{\mu\nu}$ with elements

$$(H_{\mu\nu})_{\alpha\beta} = \delta_{\alpha\mu}\delta_{\beta\nu}.$$

The set of superoperators $\hat{H}_{\mu\nu}$ is a basis (Weyl basis [60]) of Lie algebra $gl(16, \mathbb{R})$ such that

$$[\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}] = \delta_{\nu\alpha}\hat{H}_{\mu\beta} - \delta_{\mu\beta}\hat{H}_{\nu\alpha}$$

where $\mu, \nu, \alpha, \beta = 0, 1, \ldots, 15$. Any element $\hat{H}$ of the algebra $gl(16, \mathbb{C})$ can be represented by

$$\hat{H} = \sum_{\mu=0}^{15}\sum_{\nu=0}^{15} h_{\mu\nu}\hat{H}_{\mu\nu}$$

where $h_{\mu\nu}$ are complex coefficients.

As a basis of Lie algebra $gl(16, \mathbb{C})$ we can use 256 linearly independent self-adjoint superoperators

$$H_{\alpha\alpha} = |\alpha)(\alpha| \qquad H^r_{\alpha\beta} = |\alpha)(\beta| + |\beta)(\alpha|$$
$$H^i_{\alpha\beta} = -\mathrm{i}(|\alpha)(\beta| - |\beta)(\alpha|)$$

where $0 \leqslant \alpha \leqslant \beta \leqslant 15$. The matrices of these generators are Hermitian 16×16 matrices. The matrix elements of 256 Hermitian 16×16 matrices $H_{\alpha\alpha}$, $H^r_{\alpha\beta}$ and $H^i_{\alpha\beta}$ are defined by

$$(H_{\alpha\alpha})_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\alpha} \qquad \left(H^r_{\alpha\beta}\right)_{\mu\nu} = \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}$$
$$\left(H^i_{\alpha\beta}\right)_{\mu\nu} = -\,\mathrm{i}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}).$$

For any Hermitian generator $\hat{H}$ there exists a one-parameter pseudo-gate $\hat{L}(t)$ which can be represented in the form $\hat{L}(t) = \exp \mathrm{i}t\hat{H}$ such that $\hat{L}^\dagger(t)\hat{L}(t) = \hat{I}$.

Let us write the main operations which allow us to derive new pseudo-gates $\hat{L}$ from a set of pseudo-gates.

1. We introduce general SWAP (twist) pseudo-gate $\hat{T}^{(SW)}$. A new pseudo-gate $\hat{L}^{(SW)}$ defined by $\hat{L}^{(SW)} = \hat{T}^{(SW)}\hat{L}\hat{T}^{(SW)}$ is obtained directly from $\hat{L}$ by exchanging two ququats.

2. Any superoperator $\hat{L}$ on $\overline{\mathcal{H}}^{(2)}$ generated by the commutator $\mathrm{i}[\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}]$ can be obtained from $\hat{L}_{\mu\nu}(t) = \exp \mathrm{i}t\hat{H}_{\mu\nu}$ and $\hat{L}_{\alpha\beta}(t) = \exp \mathrm{i}t\hat{H}_{\alpha\beta}$ because
$$\exp t[\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}] = \lim_{n\to\infty} (\hat{L}_{\alpha\beta}(-t_n)\hat{L}_{\mu\nu}(t_n)\hat{L}_{\alpha\beta}(t_n)\hat{L}_{\mu\nu}(-t_n))^n$$
where $t_n = 1/\sqrt{n}$. Thus we can use the commutator $\mathrm{i}[\hat{H}_{\mu\nu}, \hat{H}_{\alpha\beta}]$ to generate pseudo-gates.
3. Every transformation $\hat{L}(a, b) = \exp \mathrm{i}\hat{H}(a, b)$ of $GL(16, \mathbb{C})$ generated by superoperator $\hat{H}(a, b) = a\hat{H}_{\mu\nu} + b\hat{H}_{\alpha\beta}$, where a and b are complex, can be obtained from $\hat{L}_{\mu\nu}(t) = \exp \mathrm{i}t\hat{H}_{\mu\nu}$ and $\hat{L}_{\alpha\beta}(t) = \exp \mathrm{i}t\hat{H}_{\alpha\beta}$ by
$$\exp \mathrm{i}\hat{H}(a, b) = \lim_{n\to\infty} \left(\hat{L}_{\mu\nu}\left(\frac{a}{n}\right)\hat{L}_{\alpha\beta}\left(\frac{b}{n}\right)\right)^n. \qquad \square$$

For other details of the proof, see [43, 57, 58] and [55, 56, 59].

8. Conclusion

In this paper we demonstrate a model of quantum computations with mixed states. The computations are realized by quantum operations, not necessarily unitary. Mixed states subject to the general quantum operations could increase efficiency. This increase is connected with the increasing number of computational basis elements for Hilbert space. A pure state of n two-level quantum systems is an element of 2^n-dimensional functional Hilbert space. A mixed state of the system is an element of $(2^n)^2 = 4^n$-dimensional operator Hilbert space. The conventional quantum two-valued logic is replaced by quantum four-valued logic. Therefore the increased efficiency can be formalized in terms of a four-valued logic replacing the conventional two-valued logic. Unitary gates and quantum operations for a quantum computer with pure states and two-valued logic can be considered as four-valued logic gates of a mixed state quantum computer. Quantum algorithms [72–74] on a quantum computer with mixed states are expected to run on a smaller network than with pure state implementation.

In the quantum computer model with pure states, control of quantum unitary gates is realized by classical parameters of the Hamilton operator. Open and closed quantum systems can be described by the generalized von Neumann equation [2, 37, 38]:
$$\frac{\partial}{\partial t}\rho(t) = \hat{\Lambda}\rho(t) \tag{57}$$
where $\hat{\Lambda}$ is the Liouville superoperator. For closed quantum systems this superoperator is defined by Hamiltonian H:
$$\hat{\Lambda} = -\frac{\mathrm{i}}{\hbar}(\hat{L}_H - \hat{R}_H)$$
where $\hat{L}_H$ and $\hat{R}_H$ are superoperators defined by $\hat{L}_H\rho = H\rho$ and $\hat{R}_H\rho = \rho H$. Quantum unitary gates on pure states are controlled by classical parameters entering the Hamiltonian H. For open quantum systems with completely positive evolution the Liouville superoperator $\hat{\Lambda}$ is given by
$$\hat{\Lambda} = -\frac{\mathrm{i}}{\hbar}(\hat{L}_H - \hat{R}_H) + \frac{1}{2\hbar}\sum_{j=1}^{m}\left(2\hat{L}_{V_j}\hat{R}_{V_j^\dagger} - \hat{L}_{V_j}\hat{L}_{V_j^\dagger} - \hat{R}_{V_j^\dagger}\hat{R}_{V_j}\right)$$
where H is a bounded self-adjoint Hamilton operator, $\{V_j\}$ is a sequence of bounded operators [2, 37, 38, 75–79]. Quantum four-valued logic gates on mixed states are controlled by classical parameters of the Hamiltonian H and the bounded operators V_j [21, 38].

In the paper we consider universality for general quantum four-valued logic gates acting on mixed states. The matrices of the quantum gates can be considered as elements of some matrix group but these matrices are completely positive (or positive) elements of this matrix group. The condition of complete positivity leads to difficult inequalities for matrix elements [65–68]. The completely positive condition for quantum four-valued logic gates can be satisfied by the Kraus representation (48). To find the universal set of quantum four-valued logic gates we suggest considering the universal set of superoperators (50) called pseudo-gates. Pseudo-gates are not necessarily completely positive and the set of pseudo-gates matrices is a group. In the paper we show that almost any two-ququat pseudo-gate is universal.

In the usual quantum computer model a measurement of the final pure state is described by projection operators $P_k = |k\rangle\langle k|$. In the suggested model a measurement of the final mixed state can be described by projection superoperators [32] described by $\hat{\mathcal{P}}_\mu = |\mu)(\mu|$, where $|\mu)$ are defined by (17) and (18).

A scenario for laboratory realization of quantum computations by quantum operations with mixed states can be a generalization of the scheme [80]. The quantum gates on mixed states can be realized by controlled polarization of the laser field. The control of the field polarization leads to control of the polarization mixed state of the electron. The scheme can use polarization sensitive optical fluorescence and single photon detection for read-out.

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