The Study of Isochronicity and Critical Period Bifurcations on Center Manifolds of 3-dim Polynomials Systems Using Computer Algebra

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Abstract

Using the solution of the center-focus problem from [4], we present the investigation of isochronicity and critical period bifurcations of two families of cubic 3-dim systems of ODEs. Both cubic systems have a center manifold filled with closed trajectories. The presented study is performed using computer algebra systems Mathematica and Singular.

Keywords

Polynomial systems of ODEs, center manifolds, isochronicity, bifurcation of critical periods, CAS

1 Introduction

The main topic of our work is the investigation of the quadratic 3D system of ODEs

$$\begin{aligned} \dot{u} &= -v + au^2 + av^2 + cuw + dvw, \\ \dot{v} &= u + bu^2 + bv^2 + euw + fvw, \\ \dot{w} &= -w + Su^2 + Sv^2 + Tuw + Uvw, \end{aligned} \tag{1}$$

with real coefficients a, b, c, d, e, f, S, T and U. System (1) was studied already in [4], and further in [5], [8], where planar polynomial systems of ODEs appearing on the center manifold of (1) were investigated.

We present the criteria on the coefficients of the system to distinguish between the cases of isochronous and non-isochronous oscillations, considered in [5] and [8]. Bifurcations of critical periods of the system are studied as well. Both phenomena as well as the linearization and the derivation of the period function (2) and the linearizability quantities are defined in the following section.

In order to study the period function

$$T(r) = 2\pi \left(1 + \sum_{k=1}^{\infty} T_k r^k\right) \tag{2}$$

of the centers on the center manifolds and obtain the necessary and sufficient conditions of isochronicity of the centers and to describe the critical period bifurcations (c.f. [10]) we have used the computer algebra system Mathematica and the special purpose computer algebra system Singular [7], which has powerful routines for analyzing polynomial ideals, to find the zero sets (varieties) of the obtained polynomial ideals. To obtain the corresponding ideals we used the polar coordinate approach as well as the complexification method for two dimensional polynomial systems (both explained in the following section). It turns out [10] that the isochronicity problem can be reduced to the linearizability problem, so we can reduce the problem of isochronicity to finding the variety of the ideal generated by (all) linearizability quantities, i_{kk}, j_{kk}, $k = 1, 2, \ldots$, which are of polynomial dependence on the parameters of (1). On the other hand we can consider directly the isochronicity ideal, generated by coefficients T_k (which are also of polynomial dependence on the coefficients of (1)). We denote the so called linearizability ideal (generated by all linearizability quantities i_{kk}, j_{kk}, $k = 1, 2, \ldots$) by

$$\mathcal{L} = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots \rangle \tag{3}$$

and $\mathcal{L}_K = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots, i_{KK}, j_{KK} \rangle$. To solve the problem of linearizability means to find an integer $K \geq 1$ such that $\mathbf{V}(\mathcal{L}) = \mathbf{V}(\mathcal{L}_K)$ (i.e. the variety of the linearizability ideal equals to the variety of the ideal generated by first K pairs of linearizability quantities). For this we compute the irreducible decomposition of $\mathbf{V}(\mathcal{L}_K)$ and using appropriate methods show that all systems from each component of the decomposition are linearizable (implying the obtained conditions being sufficient).

2 Definitions

The linear part of system (1) at the origin has two pure imaginary and one non-zero (real) eigenvalue. By definition a C^k-manifold $W^c \equiv W^c(0, U)$ in a neighborhood U of 0 is said to be a *center manifold* of (1) if W^c is invariant under the flow as long as the solution remains in U and W^c is the graph of a C^k-function $w = h(u, v)$ which is tangent at 0 to the (u, v)-space. There is a fundamental theorem (c.f. [2]) which implies that there exists a neighborhood U of 0 such that there exists a local center manifold W^c of (1). Note that on any local center manifold, $w = h(u, v)$, system (1) becomes a two dimensional (real) system, which can be put in the form

$$\begin{aligned} \dot{u} &= -v + P(u, v), \\ \dot{v} &= u + Q(u, v). \end{aligned} \tag{4}$$

Usually for real two dimensional polynomial systems of the form (4) with maximal degree n the qualitative analysis is done either by introducing $x = u + iv$ and $y = \overline{x} = u - iv$ and obtain the so called *complexification*

$$\dot{x} = x - \sum_{p+q=1}^{n-1} a_{p,q} x^{p+1} y^q, \qquad \dot{y} = -y + \sum_{p+q=1}^{n-1} b_{q,p} x^q y^{p+1},$$

for which the linearizability problem is to decide whether the system can be transformed to the linear system $\dot{X} = X$, $\dot{Y} = -Y$ by means of a formal change of the plane variables

$$X = x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a, b) x^m y^j, \quad Y = y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a, b) x^m y^j. \tag{5}$$

If such a transformation exists we say that the system is *linearizable*.

Differentiating with respect to t on both sides of the above two equalities and substituting the complexification in the resulted equalities and then using (5) and the original system (4) yields (after equating coefficients of the same powers) a linear recurrence system for $u^{(1)}_{m-1,j}$ and $u^{(2)}_{m,j-1}$. It turns out (see [10], p. 191) that $u^{(1)}_{q_1,q_2}$ and $u^{(2)}_{q_1,q_2}$ can be computed whenever $q_1 \neq q_2$. For $q_1 = q_2 = k \in \mathbb{N}$ some additional (polynomial) conditions, let's say $i_{kk} = 0$ and $j_{kk} = 0$ must be fulfilled. The quantities i_{kk} and j_{kk} are called *k-th linearizability quantities.* They generate the linearizability ideal defined above.

If P and Q in (4) are polynomials of degree at most n without constant and linear terms, it is convenient to introduce the polar coordinates $u = r\cos\varphi$, $v = r\sin\varphi$ and find the so-called *Poincaré return map* $\mathcal{R}(r)$, defined by the equation of the trajectories

$$\frac{dr}{d\varphi} = \frac{r^2 F(r, \cos\varphi, \sin\varphi)}{1 + rG(r, \cos\varphi, \sin\varphi)} = R(r, \varphi). \tag{6}$$

The function $R(r, \varphi)$ is periodic (with the least period 2π in variable φ) and analytic for (small enough) $|r| < r^*$ (and all φ); [8]. Thus, we can expand $R(r, \varphi)$ in a convergent power series in r to obtain

$$\frac{dr}{d\varphi} = r^2 R_2(\varphi) + r^3 R_3(\varphi) + \cdots. \tag{7}$$

One can choose (c.f. [8]) the line segment $\Sigma = \{(u,v)\,;\ v=0,\ 0\le u\le r^*\}$, where r^* is chosen to be small enough, to consider the first return of (6) from $r\,(\varphi=0)=r_0$ to $r\,(\varphi=2\pi)=\mathcal{R}\,(r_0)$.

Expanding $r\,(\varphi, r_0)$ into a (for all $\varphi\in[0,2\pi]$ and all $|r_0|\le r^*$ convergent) power series in r_0 one obtains

$$r\,(\varphi, r_0) = w_1\,(\varphi)\, r_0 + w_2\,(\varphi)\, r_0^2 + w_3\,(\varphi)\, r_0^3 + \cdots ,$$

which is a solution of (7) and inserting $r\,(\varphi, r_0)$ into (7) yields recurrence differential equations for functions $w_j\,(\varphi)$, defining the *Poncaré return map*

$$\mathcal{R}\,(r_0) := r\,(2\pi, r_0) = r_0 + w_2\,(2\pi)\, r_0^2 + w_3\,(2\pi)\, r_0^3 + \cdots .$$

Obviously, zeros of the difference function $\mathcal{P}\,(r_0) = \mathcal{R}(r_0) - r_0$ correspond to closed orbits. In particular, isolated zeros correspond to *limit cycles* and if $\mathcal{P}\,(r_0)\equiv 0$ the system has a center at the origin, yielding the conditions $w_j\,(2\pi)=0$ for all $j>1$.

Suppose the origin is center for system (4) and that the number $r^*>0$ is so small that the line segment $\Sigma = \{(u,v)\,;\ v=0,\ 0\le u\le r^*\}$ lies wholly within the period annulus. For r satisfying $0<r<r^*$, let $T\,(r)$ denote the least period of the trajectory through $(u,v)=(r,0)\in\Sigma$. The function $T\,(r)$ is the *period function* of the center. If $T\,(r)$ is constant, then the center is said to be *isochronous.* It turns out (c.f. [10], p. 176-180) that $T\,(r)$ from (2) can be written in the form

$$T(r) = 2\pi(1+\sum_{k=1}^{\infty} p_{2k} r^{2k}). \tag{8}$$

Finally, note that any value $r>0$ ($r<r^*$) for which $T'(r)=0$ is called a *critical period.* When we consider bifurcations of critical periods we are interested in an upper bound of the number of critical periods in small neighborhood of the singular point; it is the so-called problem of *critical period bifurcations,* considered for the first time in [1].

For computing the irreducible decomposition of an ideal a modular approach can be used. The Singular routine (c.f. [3]) minAssGTZ, which is based on the algorithm of [6], involves multiple computations of Gröbner bases which are extremely time and memory consuming, especially for large polynomials which is ususally the case in computations mentioned above. Thus, the routine minAssGTZ very seldom is able to complete computations and return minimal associate primes in cases of non trivial ideals (generated for instance by focus or linearizability quantities or the coefficients, T_k, of the period function (2)) when computing over the field of rational numbers. To overcome the difficulty the modular approach described in [9] has proved to be very efficient. Following the approach one first computes minimal associate primes over a field $\mathbb{Z}_p$ of a prime characteristic p (usually $p=32003$ is taken), and then lifts the obtained decomposition to the polynomial ring of characteristic zero using the rational reconstruction algorithm of [11] applied in Mathematica.

3 Main results

Edneral et al. [4] studied the dynamics of trajectories at the center manifold for the system (1). They found five conditions for the existence of a center on the center manifold:

1. $S=0$;
2. $a=b=c+f=8c+T^2-U^2=4(e-d)-T^2U^2=2(e+d)+TU=0$ and $S=1$;
3. $a=b=c=f=d+e=0$ and $S=1$;
4. $d+e=c=f=T-2a=U-2b=0$ and $S=1$;
5. $c=d=e=f=0$ and $S=1$.

In the sequel, for cases 1. and 4. (defined above) we state some results on isochronicity and critical period bifurcations of a center on the center manifold of (1).

Case 1. Obviously $w=0$ is a center manifold and the corresponding 2D system is

$$\begin{aligned} \dot u &= -v + a\left(u^2+v^2\right), \\ \dot v &= u + b\left(u^2+v^2\right). \end{aligned} \tag{9}$$

Isochronicity of (9) was studied in [8] by introducing the polar coordinates. Following the procedure described in the previous section we find that $T_2 = 2\pi\left(a^2+b^2\right)$. Thus, we see that the necessary condition for isochronicity of system (9) is $a = b = 0$, which, obviously, is also the sufficient condition. To obtain some information about critical periods of system (9) we investigate the derivative, $T'(r) = 2T_2(a,b)\,r + 3T_3(a,b)\,r^2 + \cdots$, of the period function (2). Critical periods of system (9) are zeros of $T'(r) = 0$. Recall that series (2) converges for r small enough. Note that the coefficients T_k regarded as polynomials in variables a and b are homogeneous. Since $T_2 = 2\pi\left(a^2+b^2\right) > 0$ for all (a,b) near the origin, by [10], Lemma 6.4.2, we have the following result.

Theorem 3.1. *System (9) has an isochronous center if and only if $a = b = 0$ and no critical periods bifurcate from centers of system (9).*

Case 4. On the center manifold $u^2+v^2-w=0$ (c.f. [4]) the corresponding 2D system reads

$$\begin{aligned} \dot u &= -v + (a+dv)\left(u^2+v^2\right), \\ \dot v &= u + (b-du)\left(u^2+v^2\right). \end{aligned} \tag{10}$$

The isochronicity problem and the related problem of linearizability seem to be at first glance two different problems. However, according to a theorem of Poincaré and Lyapunov (see e.g. Theorem 4.2.1 in [10]) these two problems are equivalent.

In (10) after substituting

$$a_{11} = b_{11} = d, \quad a_{01} = -b + ia \quad \text{and} \quad b_{10} = -b - ia \tag{11}$$

one obtains system

$$\begin{aligned} \dot x &= i(x - a_{11}x^2y - a_{01}xy), \\ \dot y &= -i(y + b_{11}xy^2 + b_{10}xy), \end{aligned} \tag{12}$$

where $a_{kj}, b_{kj} \in \mathbb{C}$.

We divide by i and consider a_{kj}, b_{kj} as independent parameters (not necessary satisfying condition (11)) and y as an independent unknown function (not necessary satisfying the condition $y = \overline{x}$) and solve the problem of linearizability for this more general system, obtaining the following result.

Theorem 3.2. *System (12) is linearizable if and only if one of the following conditions holds:*

1) $a_{01}b_{10} + b_{11} = b_{10} = a_{11} - b_{11} = 0$;

2) $a_{01}b_{10} + b_{11} = a_{01} = a_{11} - b_{11} = 0$.

The Darboux linearization in the proof of the above theorem (see the proof of Th. 2 in [5]) yields the following first two isochronicity quantities for real system (10):

$$\begin{aligned} p_2 &= a^2 + b^2 + d \\ p_4 &= -2(a^2+b^2)^2. \end{aligned} \tag{13}$$

Now, we obtain some information about critical periods of system (10) investigating the derivative $T'(r)$ of period function.

Theorem 3.3. *If in system* (10)

$$d = -a^2 - b^2 \tag{14}$$

then one critical period bifurcates from the origin after small perturbations.

Proof. Inserting (13) into $T'(r)$ we obtain

$$T'(r,(a,b,d)) = 2p_2(a,b,d)r + 4p_4(a,b,d)r^3 + \cdots. \tag{15}$$

Let system (10) with parameters $a = a^*$, $b = b^*$, $d = d^*$ satisfies condition (14), that is, $d^* = -a^{*2} - b^{*2}$. If $a^{*2} + b^{*2} \neq 0$, then $p_4 < 0$. Choosing $d > -a^2 - b^2$ and sufficiently small we obtain $p_2 > 0$ and $|p_2| \ll |p_4|$, yielding a system with a small root of $T'(r)$ near the origin. If $d = a = b = 0$ then we first perturb the system in such a way that $d = -a^2 - b^2$ and then apply the perturbation described above, again obtaining a critical period of the period function in a small neighborhood of the origin. □

Corollary 3.4. *System corresponding to the fourth case above has isochronous center if and only if $a = b = d = 0$.*

From the real system (10) computing we find $T_2 = a^2 + b^2 + d$ and $T_4 = -2\left(a^2 + b^2\right)^2$. By results of [10], p. 287-295, to prove that at most one critical period bifurcates from a center it is sufficient to show that $T_{2k} \in \langle T_2, T_4 \rangle$ for all $k > 2$. However, using its complex form (12) one can prove the equivalent statement, namely: $p_{2k} \in \langle p_2, p_4 \rangle$ for all $k > 2$. In [8], Th. 3.5, the following theorem is proved:

Theorem 3.5. *At most one critical period bifurcates from centers on the center manifold of system (10) after small perturbations.*

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