Orbital Reversibility of Planar Dynamical Systems

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Abstract

We give a necessary condition for the orbital-reversibility of a planar system, namely, the existence of a normal form under equivalence which is reversible to the change of sign in the first variable. Based in this condition, we formulate a suitable algorithm to detect orbital-reversibility and we apply the results to solve the center problem in a family of planar nilpotent systems.

Keywords

Bifurcation, Reversible system, Center problem, Orbital Reversible system

1 Introduction

Consider a planar autonomous system of differential equations having an equilibrium point at the origin given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{x} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$. We study if it admits some reversibility modulo $\mathcal{C}^\infty$-equivalence (see [1] and [2]).

The problem of determining if system (1.1) has some reversibility is consider in [3] and [4]. In this work, we study if there exists some time-reparametrization such that the resulting system admits some reversibility. The existence of some orbital-reversibility is a valuable feature that helps in the understanding of the dynamical behaviour of a given system.

Next, we give a precise definition of the reversibility we will deal with:

An involution is a local diffeomorphism $\sigma \in \mathcal{C}^\infty$, such that $\sigma \circ \sigma = Id$, $\sigma(\mathbf{0}) = \mathbf{0}$ and $\mathrm{codim}(\mathrm{Fix}(\sigma)) = 1$, where $\mathrm{Fix}(\sigma) = \{\mathbf{x} \in \mathbb{R}^n : \sigma(\mathbf{x}) = \mathbf{x}\}$ is the fixed point set of σ.

We say that system (1.1) is reversible if there exists some involution σ such that $\sigma_{\,*}\,\mathbf{F} = -\mathbf{F}$.

We say that system (1.1) is orbital-reversible if there exist an involution σ and a function $\mu \in \mathcal{C}^\infty$, with $\mu(\mathbf{0}) = 1$ such that $\sigma_{\,*}\,(\mu\mathbf{F}) = -\mu\mathbf{F}$, (this means that $\mathbf{F}$ is reversible modulo a time-reparametrization).

We have denoted the pull-back of a vector field of $\mathbf{F}$ by a transformation Φ as $\Phi_{\,*}\,\mathbf{F}$. If we use a generator of the transformation, the notation $\mathbf{U}_{\,**}\,\mathbf{F} := \Phi_{\,*}\,\mathbf{F}$ will be used instead. The transformed system can be expressed in terms of nested Lie products. Let us define $T_{\mathbf{U}}^{(0)}(\mathbf{F}) := \mathbf{F}$, and

$$T_{\mathbf{U}}^{(l)}(\mathbf{F}) := T_{\mathbf{U}}^{(l-1)}([\mathbf{F},\mathbf{U}]) = \overbrace{[\cdots[}^{l \text{ times}}\ \mathbf{F},\mathbf{U}\ ],\cdots,\mathbf{U}\ ] = \left[T_{\mathbf{U}}^{(l-1)}(\mathbf{F}),\mathbf{U}\right], \quad \text{for } l \geq 1.$$

If we use both, a nonlinear time-reparametrization $dt = \mu(\mathbf{x})dT$ and a near-identity transformation with generator $\mathbf{U}(\mathbf{x})$, then the transformed vector field is given by:

$$\mathbf{U}_{\,**}\,((1+\mu)\mathbf{F}) = \mathbf{U}_{\,**}\,\mathbf{F} + \mu\mathbf{F} + \mu\,[\mathbf{F},\mathbf{U}] + (\nabla\mu\cdot\mathbf{U})\mathbf{F} + \frac{1}{2!}[[\mu\mathbf{F},\mathbf{U}],\mathbf{U}] + \cdots. \tag{1.2}$$

In our study, we assume a quasi-homogeneous expansion for the vector fiel $\mathbf{F}$ corresponding to a type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$. So, we can suppose that $\mathbf{F}$ is of the form

$$\mathbf{F}(\mathbf{x}) = \tilde{\mathbf{F}}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \quad \text{for some } \ r \in \mathbb{Z}, \tag{1.3}$$

where the lowest-degree quasi-homogeneous term $\tilde{\mathbf{F}}_r \neq \mathbf{0}$ is R_x-reversible, and $\mathbf{F}_{r+k} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$ for all $k \in \mathbb{N}$.

2 Some Definitions and Main Result

In this section, we introduce some definitions and we present our important result.

Firstly, we introduce the following vector spaces:

- $\mathcal{O}^{\mathbf{t}}_k = \{\mu \in \mathcal{P}^{\mathbf{t}}_k : \mu(-x,y) = -\mu(x,y)\}$, the set of quasi-homogeneous scalar functions of degree k which are odd in the first variable.
- $\mathcal{E}^{\mathbf{t}}_k = \{\mu \in \mathcal{P}^{\mathbf{t}}_k : \mu(-x,y) = \mu(x,y)\}$, the set of quasi-homogeneous scalar functions of degree k which are even in the first variable.
- $\mathcal{R}^{\mathbf{t}}_k = \{\mathbf{F} = (P,Q)^T \in \mathcal{Q}^{\mathbf{t}}_k : P \in \mathcal{E}^{\mathbf{t}}_{k+t_1}, Q \in \mathcal{O}^{\mathbf{t}}_{k+t_2}\}$, the set of R_x–reversible quasi-homogeneous vector fields of degree k.
- $\mathcal{S}^{\mathbf{t}}_k := \{\mathbf{F} = (P,Q)^T \in \mathcal{Q}^{\mathbf{t}}_k : P \in \mathcal{O}^{\mathbf{t}}_{k+t_1}, Q \in \mathcal{E}^{\mathbf{t}}_{k+t_2}\}$, the set of R_x–symmetric quasi-homogeneous vector fields of degree k.

It is easy to deduce that $\mathcal{P}^{\mathbf{t}}_k = \mathcal{O}^{\mathbf{t}}_k \bigoplus \mathcal{E}^{\mathbf{t}}_k$ and $\mathcal{Q}^{\mathbf{t}}_k = \mathcal{R}^{\mathbf{t}}_k \bigoplus \mathcal{S}^{\mathbf{t}}_k$. This decomposition allow us to define the corresponding projection operators as follows:

$$\pi^{(\mathbf{o})}(\mu) \in \bigoplus_k \mathcal{O}^{\mathbf{t}}_k, \quad \pi^{(\mathbf{e})}(\mu) \in \bigoplus\nolimits_k \mathcal{E}^{\mathbf{t}}_k, \quad \text{for } \mu \in \bigoplus_k \mathcal{P}^{\mathbf{t}}_k, \text{ and}$$
$$\Pi^{(\mathbf{r})}(\mathbf{U}) \in \bigoplus_k \mathcal{R}^{\mathbf{t}}_k, \quad \Pi^{(\mathbf{s})}(\mathbf{U}) \in \bigoplus\nolimits_k \mathcal{S}^{\mathbf{t}}_k, \quad \text{for } \mathbf{U} \in \bigoplus_k \mathcal{Q}^{\mathbf{t}}_k.$$

The main goal of this paper is to determine conditions for the orbital–reversibility of (1.3), which will be based on the existence of a near-identity transformation $\Phi = \sum_{j\geq 0}\Phi_j$, $(\Phi_j \in \mathcal{Q}^{\mathbf{t}}_j)$, and a scalar function $\mu \in \mathcal{C}^\infty$, with $\mu(\mathbf{0}) = 1$, such that $\Phi_*(\mu\mathbf{F})$ is R_x–reversible.

For our convenience, from now on we will write the time-reparametrization as $1+\mu$, with $\mu(\mathbf{0}) = 0$. Indeed, it will be written as $1 + \sum_{j\geq 1}\mu_j$, where $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$ for $j \geq 1$.

Definition 1 *We say that the vector field of system (1.3) is* N-orbital–reversible *($N \in \mathbb{N}$) if there exist a vector field $\mathbf{U} \in \bigoplus_{j\geq 1}\mathcal{Q}^{\mathbf{t}}_j$ and a scalar function $\mu \in \bigoplus_{j\geq 1}\mathcal{P}^{\mathbf{t}}_j$, such that $\mathcal{J}^{r+N}(\mathbf{U}_{**}((1+\mu)\mathbf{F}))$ is R_x-reversible.*

Our idea is to adapt the normal form procedure in order to determine conditions under which the normalized vector field is N-orbital–reversible. We introduce the Lie derivate along the lowest-degree quasi-homogeneous term $\tilde{\mathbf{F}}_r$:

$$\begin{aligned} \ell_{k-r} \ : \ & \mathcal{P}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_k \\ & \mu_{k-r} \longrightarrow \nabla\mu_{k-r}\cdot\tilde{\mathbf{F}}_r. \end{aligned}$$

In the normal form reduction it is enough to take its quasi-homogeneous terms μ_k belonging to $\mathrm{Cor}(\ell_{k-r})$ (a complementary subspace to $\mathrm{Range}(\ell_{k-r})$).

We denote

$$\widehat{\mathcal{R}}^{\mathbf{t}}_k := \mathcal{R}^{\mathbf{t}}_k \cap \widehat{\mathcal{Q}}^{\mathbf{t}}_k \ \text{ and } \ \widehat{\mathcal{O}}^{\mathbf{t}}_k := \mathcal{O}^{\mathbf{t}}_k \cap \mathrm{Cor}(\ell_{k-r}),$$

where $\widehat{\mathcal{Q}}^{\mathbf{t}}_k$ is a complementary subspace to $\mathrm{Ker}(\ell_{k-r})\tilde{\mathbf{F}}_r$ in $\mathcal{Q}^{\mathbf{t}}_k$.

Next, we plain to deduce some facts about the normal forms for orbital-reversible vector fields. To this end, we use that $\mathcal{Q}^{\mathbf{t}}_k = \mathcal{R}^{\mathbf{t}}_k \bigoplus \mathcal{S}^{\mathbf{t}}_k$, which allows to write the vector field (1.3) as:

$$\mathbf{F} = \tilde{\mathbf{F}}_r + \sum_{j=1}^{\infty}(\tilde{\mathbf{F}}_{r+j} + \bar{\mathbf{F}}_{r+j}), \tag{2.4}$$

where $\tilde{\mathbf{F}}_{r+j} = \Pi^{(\mathbf{r})}(\mathbf{F}_{r+j}) \in \mathcal{R}^{\mathbf{t}}_{r+j}$ and $\bar{\mathbf{F}}_{r+j} = \Pi^{(\mathbf{s})}(\mathbf{F}_{r+j}) \in \mathcal{S}^{\mathbf{t}}_{r+j}$.

To describe a normal form procedure well adapted to the orbital-reversibility problem, let us denote the above vector field as

$$\mathbf{F}^{(0)} := \mathbf{F} = \tilde{\mathbf{F}}^{(0)}_r + (\tilde{\mathbf{F}}^{(0)}_{r+1} + \bar{\mathbf{F}}^{(0)}_{r+1}) + \cdots.$$

We observe that the lowest-degree quasi-homogeneous term is reversible: $\tilde{\mathbf{F}}_r^{(0)} \in \mathcal{R}_r^{\mathbf{t}}$.

We define the homological operator $\overline{\mathcal{L}}^{(m)}$ as,

$$\begin{array}{rcl} \overline{\mathcal{L}}^{(1)} : \widehat{\mathcal{R}}_1^{\mathbf{t}} \times \widehat{\mathcal{O}}_1^{\mathbf{t}} & \longrightarrow & \mathcal{S}_{r+1}^{\mathbf{t}} \\ (\tilde{\mathbf{U}}_1, \tilde{\mu}_1) & \longrightarrow & -[\tilde{\mathbf{F}}_r^{(0)}, \tilde{\mathbf{U}}_1] - \tilde{\mu}_1 \tilde{\mathbf{F}}_r^{(0)}, \end{array}$$

and

$$\begin{array}{rcl} \overline{\mathcal{L}}^{(m)} : \mathrm{Ker}(\overline{\mathcal{L}}^{(m-1)}) \times (\widehat{\mathcal{R}}_m^{\mathbf{t}}, \widehat{\mathcal{O}}_m^{\mathbf{t}}) & \longrightarrow & \mathcal{S}_{r+m}^{\mathbf{t}} \\ (\tilde{\mathbf{U}}_1, \tilde{\mu}_1, \cdots, \tilde{\mathbf{U}}_{m-1}, \tilde{\mu}_{m-1}; \tilde{\mathbf{U}}_m, \tilde{\mu}_m) & \longrightarrow & -\sum_{j=0}^{m-1} [\tilde{\mathbf{F}}_{r+j}^{(m-1)}, \tilde{\mathbf{U}}_{m-j}] - \tilde{\mu}_{m-j} \tilde{\mathbf{F}}_{r+j}^{(m-1)}. \end{array}$$

It is evident that operator $\overline{\mathcal{L}}^{(m)}$ depends on $\tilde{\mathbf{F}}_r^{(m)}, \cdots, \tilde{\mathbf{F}}_{r+m-1}^{(m)}$.

The following result characterizes the $(N+1)$-orbital–reversibility of a vector field N-orbital–reversible. Proceeding degree by degree and following the ideas of the classical normal form theory, we obtain an algorithm to discarding cases the orbital–reversibility based of the next theorem.

Theorem 2 *Let us consider a vector field* $\mathbf{F} = \tilde{\mathbf{F}}_r + \cdots + \tilde{\mathbf{F}}_{r+N-1} + (\tilde{\mathbf{F}}_{r+N} + \bar{\mathbf{F}}_{r+N}) + \cdots$, *satisfying* $\bar{\mathbf{F}}_{r+N} \neq 0$ *and* $\mathrm{Proj}_{\mathrm{Im}(\overline{\mathcal{L}}^{(N)})}(\bar{\mathbf{F}}_{r+N}) = \mathbf{0}$, *for some* $N \in \mathbb{N}$. *Then,* $\mathbf{F}$ *is not orbital–reversible.*

3 Application

Let us consider the following family of planar vector fields:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \sigma x^{4q+1} \end{pmatrix} + \begin{pmatrix} a_1 xy + a_2 x^{2q+2} \\ b_1 y^2 + b_2 x^{2q+1} y \end{pmatrix}, \tag{3.5}$$

where $\sigma = \pm 1$, $q \in \mathbb{N}$.

This family has been studied by several authors. Namely, the analytic integrability for this family has been studied in [5]; the center problem for $\sigma = -1$ (which corresponds to the monodromic situation) has been partially studied in [6]; and the reversibility problem is completely solved in [3]. With respect to the orbital-reversibility problem, we have the following result:

Theorem 3 *System (3.5) is orbital-reversible if and only if one of the following conditions is satisfied:*

(a) $a_2 = b_2 = 0$.

(b) $a_2 = a_1 = b_1 = 0$, $b_2 \neq 0$.

(c) $a_1 = b_1 = 0$, $a_2 \neq 0$.

(d) $a_1 + 2b_1 = b_2 + 2(q+1)a_2 = 0$, $a_2 b_1 \neq 0$.

(e) $b_2 = (2q+1)a_2$, $b_1 = (2q+1)a_1$, $a_2(a_1 + 2b_1) \neq 0$.

Proof:

The vector field of the statement can be written as $\mathbf{F} = \tilde{\mathbf{F}}_r + \mathbf{F}_{r+1}$, where

$$\tilde{\mathbf{F}}_r := (y, \sigma x^{4q+1})^T \in \mathcal{Q}_{2q}^{\mathbf{t}}, \text{ and } \mathbf{F}_{r+1} \in \mathcal{Q}_{2q+1}^{\mathbf{t}},$$

being $r = 2q$ and $\mathbf{t} = (1, 2q+1)$. We observe that $\tilde{\mathbf{F}}_{2q}$ is R_x– and R_y–reversible. It is enough to study the R_x– and the R_y–orbital-reversibility of the vector field $\mathbf{F}$.

($\star$) We start with the R_x–orbital-reversibility. As we will see later, in this case is sufficient to reach the $N = 8$-orbital–reversibility to solve the orbital–reversibility problem. To reduce the vector field of the statement to the normal form $\mathbf{F}^{(8)}$, we take the generator

$$\tilde{\mathbf{U}} = \begin{pmatrix} \alpha_1 x^2 \\ \alpha_2 xy \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_3 x^{2q+3} \end{pmatrix} + \begin{pmatrix} \alpha_4 x^4 \\ \alpha_5 x^3 y \end{pmatrix} + \cdots \in \bigoplus_{j=1}^{8} \mathcal{R}_j^{\mathbf{t}},$$

and the time-reparametrization associated to

$$\tilde{\mu} = \gamma_1 x + \gamma_3 x^3 + \gamma_5 x^5 + \gamma_7 x^7 \in \bigoplus_{j\geq 1}^{8} \widehat{\mathcal{O}}_j^{\mathbf{t}},$$

where α_i and γ_i are arbitrary parameters. Using *Maple* in the computations, we obtain the following normal form:

$$\begin{aligned}
\mathbf{F}^{(8)} &= \tilde{\mathbf{U}}_{**}((1+\tilde{\mu})\mathbf{F}) = \tilde{\mathbf{F}}_{2q} + \tilde{\mathbf{F}}_{2q+1} + (\tilde{\mathbf{F}}_{2q+2} - \frac{1}{3(4q+3)}\lambda^{(2)}\begin{pmatrix} 0 \\ x^{2q+2}y \end{pmatrix}) \\
&+\tilde{\mathbf{F}}_{2q+3} + (\tilde{\mathbf{F}}_{2q+4} + \frac{\sigma}{3(4q+5)(4q+3)^3}\lambda^{(4)}\begin{pmatrix} 0 \\ x^{2q+4}y \end{pmatrix}) \\
&+\tilde{\mathbf{F}}_{2q+5} + (\tilde{\mathbf{F}}_{2q+6} + \lambda^{(6)}\begin{pmatrix} 0 \\ x^{2q+6}y \end{pmatrix}) \\
&+\tilde{\mathbf{F}}_{2q+7} + (\tilde{\mathbf{F}}_{2q+8} + \lambda^{(8)}\begin{pmatrix} 0 \\ x^{2q+8}y \end{pmatrix}) + \cdots.
\end{aligned}$$

So, by applying Theorem 2, if $\mathbf{F}$ is orbital-reversible then the coefficients $\lambda^{(2j)}$ must vanish.

The first normal form coefficient $\lambda^{(2j)}$ is:

$$\lambda^{(2)} = a_2((2q+3)(2q+1)a_1 + 2qb_1) + b_2(2qa_1 - 3b_1). \tag{3.6}$$

To study the vanishing of this coefficient, we consider the following two possibilities:

(1) $2qa_1 - 3b_1 = 0$, and then $\lambda^{(2)}$ vanishes in a couple of cases:

(1a) $a_2 = 0$. In this case, the next normal form coefficient is

$$\lambda^{(4)} = qb_2a_1^3,$$

which vanishes if $b_2 = 0$ (in this case, covered in item **(a)**, the system is R_y-reversible), or if $a_1 = 0$ (now, the system is R_x-reversible; this situation is described in item **(b)**).

(1b) $a_2 \neq 0$, $(2q+3)(2q+1)a_1 + 2qb_1 = 0$, which provides $a_1 = b_1 = 0$. In this case the system is R_x-reversible. This is the situation described in item **(c)**.

(2) $2qa_1 - 3b_1 \neq 0$, and then $\lambda^{(2)}$ vanishes if, and only if,

$$b_2 = -\frac{(2q+3)(2q+1)a_1 + 2qb_1}{2qa_1 - 3b_1}a_2. \tag{3.7}$$

For this value, the next normal form coefficient is

$$\lambda^{(4)} = \frac{4q+3}{2qa_1 - 3b_1}a_2(a_1 + 2b_1)(b_1 - (2q+1)a_1)p_4(a_2, a_1, b_1, q, \sigma),$$

where we have denoted

$$\begin{aligned}
&p_4(a_2, a_1, b_1, q, \sigma) = 3(2q+5)(4q+3)^2((2q+3)(4q+1)a_1 - (4q+9)b_1)a_2^2 \\
&\quad +\sigma(2qa_1 - 3b_1)(2q(120q^2 + 202q + 49)a_1^2 - (512q^2 + 844q + 135)a_1b_1 + 5(52q+81)b_1^2).
\end{aligned}$$

The vanishing of $\lambda^{(4)}$ leads to some subcases:

(2a) $a_2 = 0$, which implies $b_2 = 0$. We get again item **(a)**.

(2b) $a_2 \neq 0$, $a_1 + 2b_1 = 0$. This hypothesis implies that $b_1 \neq 0$ (otherwise, $a_1 = b_1 = 0$). Moreover, the equation (3.7) reduces to $b_2 = -2(q+1)a_2$. Now, the system (3.5) is Hamiltonian, with Hamiltonian

$$h(x,y) = -\frac{1}{2}y^2 + \frac{\sigma}{2(2q+1)}x^{4q+2} + b_1xy^2 - a_2x^{2q+2}y.$$

If we denote $u = x$, $v = y - 2b_1xy + a_2x^{2q+2}$, then system (3.5) becomes:

$$\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= \sigma u^{4q+4} + (2(q+1)a_2^2 - 2b_1\sigma)u^{4q+2} + \frac{a_2u^{4q+4} - b_1v^2}{1 - 2b_1u},
\end{aligned}$$

which is R_v–reversible (item **(d)**).

(2c) $a_2(a_1+2b_1) \neq 0$, $b_1 = (2q+1)a_1$. Now, the equation (3.7) reduces to $b_2 = (2q+1)a_2$.

In this case, it is more convenient to work with system (3.5) with the transformation $x = u$, $y = v(1+a_1u)^{2q+1}$, i.e.:

$$\begin{aligned} \dot{u} &= v(1+a_1u)^{2q+2} + a_2u^{2q+2}, \\ \dot{v} &= \frac{\sigma u^{4q+1}}{(1+a_1u)^{2q+1}} + \frac{(2q+1)a_2}{1+a_1u}u^{2q+1}v. \end{aligned}$$

The time reparametrization $dT = (1+a_1X)^{2q}dt$ and the transformation $X = \frac{u}{1+a_1u}$, $Y = v$, yield

$$\begin{aligned} X' &= Y + a_2X^{2q+2}, \\ Y' &= \sigma X^{4q+1} + (2q+1)a_2X^{2q+1}Y, \end{aligned}$$

which is R_X–reversible (item **(e)**).

(2d) $a_2(a_1+2b_1)(b_1-(2q+1)a_1) \neq 0$, $p_4(a_2,a_1,b_1,q,\sigma) = 0$. In this case, both coefficients $\lambda^{(6)}$ and $\lambda^{(8)}$ can not vanish simultaneously, and the vector field is not orbital-reversible.

($\star\star$) The situation with the R_y–orbital-reversibility does not include any new case.

From the proof of the theorem, we obtain that system (3.5) is orbital-reversible if, and only if, it is 8-orbital reversible.

References

[1] J.S.W. Lamb and J.A.G. Roberts. *Time-reversal symmetry in dynamical systems: a survey.* Physica D. Nonlinear Phenomena, 112, 1-2, (1998), pp. 1-39.

[2] D. Montgomery and L. Zippin. *Topological Transformations Group.* Interscience Publ., New York, 1995.

[3] A. Algaba, C. García and M.A. Teixeira. *Reversibility and quasi-homogeneous normal forms of vector fields.* Nonlinear Analysis: Theory, Methods and Applications, 73, (2010), pp. 510-525.

[4] A. Algaba, E. Gamero and C. García. *The reversibility problem for quasi-homogeneous dynamical systems.* Discrete and Continuous Dynamical System-A, 33, 8, (2013), pp. 3225-3236. doi:10.3934/dcds.2013.33.3225

[5] J. Chavarriga, I. García and J. Giné. *Integrability of centers perturbed by quasi-homogeneous polynomials.* Journal of Mathematical Analysis and Applications, 210 (1997), pp. 268–278.

[6] A. Gasull and J. Torregrosa. *Center problem for several differential equations via Cherkas' method.* Journal of Mathematical Analysis and Applications, 228 (1998), pp. 322–343.