On Using of Computer Algebra Systems for Analysis of Rigid Body Dynamics

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Abstract

The paper presents some results of qualitative analysis of conservative systems. The modified Routh-Lyapunov technique is used as tool for investigation. Special attention is paid to algorithms of finding and analysis of invariant manifolds on which elements of algebra of problem's first integrals assume a stationary value.

Keywords
first integrals, invariant manifolds, conservative system, stability

Introduction

Application of modern tools of computer algebra (CA) allows one significantly to increase the number of effective algorithms which are used for qualitative analysis of dynamic systems. The paper discusses several algorithms which are some generalization of the Routh-Lyapunov technique [1] of analysis of conservative systems with algebraic first integrals. These algorithms are: the use of enveloping integral for family of first integrals in order to find invariant manifolds (IM) and to investigate their stability [2]; solving a system of stationary equations of a family of first integrals with respect to some part of phase variables and some part of parameters of family's first integrals [3]; finding IM of 2nd and higher level on earlier found IMs. Efficiency of these approaches is demonstrated by examples of analysis of two classical completely integrable systems.

1 Kovalevskaya's Case.

In Kovalevskaya's problem [4] of motion of a rigid body with a fixed point the equations of motion write

$$2\dot{p} = qr, \quad 2\dot{q} = -rp + x_0\gamma_3, \quad \dot{r} = -x_0\gamma_2, \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \dot{\gamma}_3 = q\gamma_1 - p\gamma_2,$$

and have the following first integrals

$$2H = 2p^2 + 2q^2 + r^2 + 2x_0\gamma_1 = 2h, \quad V_1 = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 = m,$$

$$V_2 = (p^2 - q^2 - x_0\gamma_1)^2 + (2p\,q - x_0\gamma_2)^2 = k^2, \quad V_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

Consider the problem of finding IMs on which Kovalevskaya's integral V_2 assumes a stationary value. The necessary conditions of extremum for integral V_2 have the form:

$$\frac{\partial V_2}{\partial p} = 4(py_1 + qy_2) = 0, \ \frac{\partial V_2}{\partial \gamma_1} = -2x_0y_1 = 0, \ \frac{\partial V_2}{\partial q} = -4(qy_1 - py_2) = 0, \ \frac{\partial V_2}{\partial \gamma_2} = -2x_0y_2 = 0. \quad (1)$$

From equations (1), where the following denotations $y_1 = p^2 - q^2 - x_0\gamma_1, \ \ y_2 = 2p\,q - x_0\gamma_2$ were used, we conclude that the equations for one of invariant manifolds of stationary motions (IMSM), which correspond to integral V_2, can be written as

$$y_1 = p^2 - q^2 - x_0\gamma_1 = 0, \quad y_2 = 2p\,q - x_0\gamma_2 = 0. \quad (2)$$

It is Delaunay's manifold. The vector field on IMSM (2) is defined by the equations:

$$2\dot{p} = qr, \quad 2\dot{q} = -r\,p + x_0\gamma_3, \quad \dot{r} = -2p\,q, \quad \dot{\gamma_3} = -q(p^2+q^2)x_0^{-1}. \tag{3}$$

Differential equations (3) have the following first integrals:

$$2\tilde{H} = 4p^2 + r^2 = 2h, \quad \tilde{V_1} = r\gamma_3 + 2p(p^2+q^2)x_0^{-1} = m, \quad \tilde{V_3} = \gamma_3^2 + (p^2+q^2)^2 x_0^{-2} = 1. \tag{4}$$

Let us state the problem of finding IMs of 2nd level on which the elements of algebra of the first integrals of system (3) assume a stationary value. To this end, we construct the following linear combination of integrals (4)

$$2\tilde{K} = 2\tilde{H} - 2\nu_1\tilde{V_1} + \nu_1^2\tilde{V_2}. \tag{5}$$

The conditions of stationarity for $\tilde{K}$ write

$$\frac{\partial\tilde{K}}{\partial p} = 2(1-\frac{\nu_1}{x_0}p)(2p-\frac{\nu_1}{x_0}(p^2+q^2)) = 0, \quad \frac{\partial\tilde{K}}{\partial q} = -\frac{2\nu_1 q}{x_0}(2p-\frac{\nu_1}{x_0}(p^2+q^2)) = 0,$$

$$\frac{\partial\tilde{K}}{\partial r} = r - \nu_1\gamma_3 = 0, \quad \frac{\partial\tilde{K}}{\partial\gamma_3} = -\nu_1(r-\nu_1\gamma_3) = 0.$$

One of degenerated families of solutions of the above system is defined by the equations:

$$2\nu_1 x_0 p - \nu_1^2(p^2+q^2) = 0, \quad r - \nu_1\gamma_3 = 0. \tag{6}$$

These are the equations of the family of IMSM on IMSM (2). The family of 2nd level IMSMs (6) can be "lifted up" as invariant into the initial phase space. To this end, it is necessary to add the Delaunay IMSM equations (2) to equations (6).

1.1 Kovalevskaya's Case. Enveloping Integral

In order to find peculiar IMSMs of 2nd level of system (3) let us apply enveloping integral for the family of integrals (5). Following to standard algorithm, we calculate derivative of integral $\tilde{K}$ with respect to parameter ν_1 (the parameter of the family of integrals) and equate the obtained result to zero:

$$\frac{\partial\tilde{K}}{\partial\nu_1} = -\tilde{V_1} + \nu_1\tilde{V_3} = 0.$$

From the latter expression we find $\nu_1 = \tilde{V_1}\tilde{V_3}^{-1}$. Consequently, the enveloping first integral of our interest has the form: $2\tilde{\tilde{K}}_0 = 2\tilde{K} - \tilde{V_1}^2\tilde{V_3}^{-1}$ or $2\tilde{\tilde{K}}_0 = 2\tilde{H}\tilde{V_3} - \tilde{V_1}^2$.

Next, write down the necessary conditions of extremum for the integral $\tilde{\tilde{K}}_0$:

$$\frac{\partial\tilde{\tilde{K}}_0}{\partial p} = 4p(\gamma_3^2 + (p^2+q^2)x_0^{-2}) + 4p(2p^2+\frac{r^2}{2})(p^2+q^2)x_0^{-2} -$$

$$2(r\gamma_3 + 2px_0^{-1}(p^2+q^2))(3p^2+q^2)x_0^{-1} = 0.$$

$$\frac{\partial\tilde{\tilde{K}}_0}{\partial q} = 4q(p^2+q^2)x_0^{-2}(2p^2+\frac{r^2}{2}) - 4pq(r\gamma_3 + 2px_0^{-1}(p^2+q^2))x_0^{-1} = 0,$$

$$\frac{\partial\tilde{\tilde{K}}_0}{\partial r} = r(\gamma_3^2 + (p^2+q^2)x_0^{-2}) + (r\gamma_3 + 2px_0^{-1}(p^2+q^2))\gamma_3 = 0,$$

$$\frac{\partial\tilde{\tilde{K}}_0}{\partial\gamma_3} = 2\gamma_3(2p^2+\frac{r^2}{2}) - (r\gamma_3 + 2px_0^{-1}(p^2+q^2))r = 0.$$

It can easily be verified that equation

$$(p^2+q^2)r - 2px_0\gamma_3 = 0 \tag{7}$$

defines IMSM on which the enveloping integral assumes a stationary value, besides this IMSM is the first integral of equations (3). The 2nd level IMSM obtained by the above method can be "lifted up" into the phase space of the initial system. To this end, likewise above, we add the equation of Delaunay's IMSM to equation (7).

1.2 Kovalevskaya's Case. Stability.

Now let us consider the problem of stability for some above obtained IMSMs.

1. Let us write down the equations of perturbed motion in the neighborhood of Delaunay's IM (2):

$$\dot{y}_1 = ry_2, \quad \dot{y}_2 = -ry_1, \quad 2\dot{p} = qr, \quad 2\dot{q} = -rp + x_0\gamma_3,$$
$$\dot{r} = y_2 - 2pq, \quad x_0\dot{\gamma_3} = -q(p^2 - q^2) + py_2 - qy_1.$$

Here $y_1 = p^2 - q^2 - x_0\gamma_1, \quad y_2 = 2p\,q - x_0\gamma_2$ are the deviations from Delaunay's IM in perturbed motion.

The system has the sign definite first integral

$$\Delta V_2 = y_1^2 + y_2^2 >> 0.$$

The latter guaranties stability of IMSM (2).

2. Next, let us consider the family of IMSMs (6). Introduce the deviations from the elements of this family of IMSMs:

$$z_1 = 2x_0 p - \nu_1(p^2 + q^2), \quad z_2 = r - \nu_1\gamma_3,$$

and write down differential equations of perturbed motion in this case. Because the first equation of IM is nonlinear, we use maps on the IMSMs. It is possible to take, for example, the following four maps when $x_0\nu_1 > 0$:

$$q = \pm\sqrt{2px_0/\nu_1 - p^2}, r = \nu_1\gamma_3, \quad (0 < p < 2x_0/\nu_1, -\nu_1 < r < \nu_1),$$

$$p = x_0/\nu_1 \pm \sqrt{x_0^2/\nu_1^2 - q^2}, \quad r = \nu_1\gamma_3, \quad (-x_0/\nu_1 < q < x_0/\nu_1, \; -\nu_1 < r < \nu_1).$$

Analogous maps can be constructed when $x_0\nu_1 < 0$. A vector field is defined in each map.

Let us write down equations of perturbed motion in the neighborhood of IM (6). In 4th map these equations have the form:

$$\dot{z}_1 = x_0 q z_2, \quad \dot{z}_2 = -qz_1/x_0, \quad \dot{r} = -2q(x_0/\nu_1 - \sqrt{x_0^2/\nu_1^2 - q^2 - z_1/\nu_1}),$$
$$2\dot{q} = -z_2 x_0/\nu_1 + r\sqrt{x_0^2/\nu_1^2 - q^2 - z_1/\nu_1}. \tag{8}$$

Analogous equations can also be written in other maps on IM (6). Equations (8) admit the first integral:

$$2\Delta K = z_2^2 + z_1^2/x_0^2.$$

In other maps the integral for equations of perturbed motion has analogous form. Because the integral is sign definite on z_1, z_2, we conclude that IMSM (6) is stable.

2 Kirchhoff's Problem

Let us consider the problem of motion of a rigid body in ideal fluid in case [5]. The differential equations of motion

$$\dot{r_1} = (\alpha r_1 + \beta r_2 + 2s_3)r_2 - r_3 s_2, \quad \dot{r_2} = -(\alpha r_1 + \beta r_2 + 2s_3)r_1 - r_3 s_1, \quad \dot{r_3} = r_1 s_2 - r_2 s_1,$$
$$\dot{s_1} = -(\beta s_3 + (\alpha^2 + \beta^2)r_2)r_3 + (\alpha r_1 + \beta r_2 + s_3)s_2,$$
$$\dot{s_2} = (\alpha s_3 + (\alpha^2 + \beta^2)r_1)r_3 - (\alpha r_1 + \beta r_2 + s_3)s_1, \; \dot{s_3} = (\beta r_1 - \alpha r_2)s_3 \tag{9}$$

admit the following first integrals:

$$2H = (s_1^2 + s_2^2 + 2s_3^2) + 2(\alpha r_1 + \beta r_2)s_3 - (\alpha^2 + \beta^2)r_3^2 = 2h,$$
$$V_1 = s_1 r_1 + s_2 r_2 + s_3 r_3 = c_1, \quad 2V_2 = r_1^2 + r_2^2 + r_3^2 = c_2,$$
$$2V_3 = (r_1 s_1 + r_2 s_2)((\alpha^2 + \beta^2)(r_1 s_1 + r_2 s_2) + 2(\alpha s_1 + \beta s_2)s_3)$$
$$+ s_3^2(s_1^2 + s_2^2 + (\alpha r_1 + \beta r_2 + s_3)^2) = 2c_3. \tag{10}$$

In order to find stationary solutions and IMSMs of system (9) we construct the families K of first integrals

$$K = \lambda_0 H - \lambda_1 V_1 - \lambda_2 V_2 - \lambda_3 V_3. \tag{11}$$

from problem's first integrals (10).

The necessary conditions of extremum for K (11) with respect to variables $s_1, s_2, s_3, r_1, r_2, r_3$

$$\begin{aligned}
\frac{\partial K}{\partial s_1} &= \lambda_0 s_1 - \lambda_1 r_1 - \lambda_3[(\alpha^2+\beta^2)r_1(r_1 s_1 + r_2 s_2) + s_2 s_3(\alpha r_2 + \beta r_1) + s_1 s_3(2\alpha r_1 + s_3)] = 0,\\
\frac{\partial K}{\partial s_2} &= \lambda_0 s_2 - \lambda_1 r_2 - \lambda_3[(\alpha^2+\beta^2)r_2(r_1 s_1 + r_2 s_2) + s_1 s_3(\alpha r_2 + \beta r_1) + s_2 s_3(2\beta r_2 + s_3)] = 0,\\
\frac{\partial K}{\partial s_3} &= \lambda_0(\alpha r_1 + \beta r_2 + 2 s_3) - \lambda_1 r_3 - \lambda_3[(\alpha s_1 + \beta s_2)(r_1 s_1 + r_2 s_2) +\\
& s_3((\alpha r_1 + \beta r_2 + 2 s_3)^2 + s_1^2 + s_2^2 - s_3(\alpha r_1 + \beta r_2 + 2 s_3))] = 0,\\
\frac{\partial K}{\partial r_1} &= \lambda_0 \alpha s_3 - \lambda_1 s_1 - \lambda_2 r_1 - \lambda_3[(\alpha^2+\beta^2)s_1(r_1 s_1 + r_2 s_2) + s_1 s_3(\alpha s_1 + \beta s_2)\\
& + \alpha s_3^2(\alpha r_1 + \beta r_2) + \alpha s_3^3 = 0,\\
\frac{\partial K}{\partial r_2} &= \beta\lambda_0 s_3 - \lambda_1 s_2 - \lambda_2 r_2 - \lambda_3[(\alpha^2+\beta^2)(r_1 s_1 + r_2) s_2 + (\alpha s_1 + \beta s_2) s_2 s_3\\
& + \beta s_3^2(\alpha r_1 + \beta r_2) + \beta s_3^3] = 0,\\
\frac{\partial K}{\partial r_3} &= -((\alpha^2+\beta^2)\lambda_0 + \lambda_2) r_3 - \lambda_1 s_3 = 0.
\end{aligned} \tag{12}$$

define the families of stationary solutions and the families of IMSM of differential equations (9). Computer algebra system MATHEMATICA allows one to apply the Gröbner basis technique [6] for finding solutions of nonlinear algebraic system. The Gröbner basis for system (12) constructed with respect to some part of parameters $\lambda_0, \lambda_1, \lambda_2$ and some part of phase variables r_3, s_3 writes:

$$\begin{aligned}
&\{\lambda_2\left(pz^2\lambda_2 + q^2x^2\lambda_3\right),\ -q^2x\lambda_1 - z\left((\beta r_1 + \alpha r_2)s_1^2 + 2\left(-\alpha r_1 + \beta r_2\right)s_1 s_2 - (\beta r_1 + \alpha r_2)s_2^2\right)\\
&\lambda_2 - Gq^2x^2\lambda_3,\ -pq^2\lambda_0 - \left(\beta^2 r_1^4 - 2\alpha\beta r_1^3 r_2 + r_2^2\left(\alpha^2 r_2^2 + s_1^2\right) - 2r_1\left(\alpha\beta r_2^3 + r_2 s_1 s_2\right) +\right.\\
&\left. r_1^2\left(Gr_2^2 + s_2^2\right)\right)\lambda_2,\ -yz\lambda_2 - q^2 x s_3\lambda_3,\ -pz\left(\alpha r_1 + \beta r_2\right)\lambda_2 + qx^2\left(Gr_3 - \alpha s_1 - \beta s_2\right)\lambda_3\}.
\end{aligned} \tag{13}$$

Here the following denotations

$$q = \beta s_1 - \alpha s_2,\ x = r_1 s_1 + r_2 s_2,\ y = r_1 s_2 - r_2 s_1,\ z = \beta r_1 - \alpha r_2,\ G = \alpha^2 + \beta^2,\ \ p = r_1^2 + r_2^2. \tag{14}$$

were used.

Let us consider one family of solutions of system (13) (here λ_3 is the family parameter):

$$\begin{aligned}
&s_3 = xy/pz,\ r_3 = y/z,\ \lambda_2 = -q^2x^2\lambda_3/pz^2,\\
&\lambda_1 = -\left(x\left(-pq^2 + Gy^2 + Gpz^2\right)\lambda_3\right)/pz^2,\ \lambda_0 = x^2\left(y^2 + pz^2\right)\lambda_3/p^2z^2.
\end{aligned} \tag{15}$$

Analysis of the above relations showed that expressions for r_3, s_3 (15) define IMSM of differential equations (9). The vector field on IMSM (15) is described by equations

$$\begin{aligned}
\dot r_1 = r_2\left(\frac{2xy}{pz} + \alpha r_1 + \beta r_2\right) - \frac{ys_2}{z},\ \dot r_2 = -r_1\left(\frac{2xy}{pz} + \alpha r_1 + \beta r_2\right) + \frac{ys_1}{z},\\
\dot s_1 = \frac{-y\left(xy\beta + Gpzr_2\right) + z\left(xy + pz\left(\alpha r_1 + \beta r_2\right)\right)s_2}{pz^2},\\
\dot s_2 = \frac{y\left(xy\alpha + Gpzr_1\right) - z\left(xy + pz\left(\alpha r_1 + \beta r_2\right)\right)s_1}{pz^2}.
\end{aligned} \tag{16}$$

The expressions $\lambda_0, \lambda_1, \lambda_2$ (15) are the first integrals of equations (16). It can be showed that these integrals correspond to the integrals of initial differential equations (9):

$$\tilde\lambda_0 = (V_1(HV_1 \pm \sqrt{(v_1^2(H^2 - 2V_3) + 8GV_2^2V_3)\lambda_3}))/(V_1^2 - 4GV_2^2),$$

$$\tilde\lambda_1 = ((2GHV_1V_2^2 \pm (V_1^2 - 2GV_2^2)\sqrt{(H^2v_1^2 - 2V_1^2V_3 + 8GV_2^2V_3)\lambda_3})/(V_2(V_1^2 - 4GV_2^2)),$$

$$\tilde\lambda_2 = (V_1^2(4GHV_2^2 \pm V_1\sqrt{(v_1^2(H^2 - 2V_3) + 8GV_2^2V_3)\lambda_3}))/(V_1^2 - 4GV_2^2).$$

2.1 Second Level Invariant Manifolds

Let us find IMSMs of 2nd level on IM (15). For this purpose, we shall use narrowing of the integral K on IMSM (15). First integrals (10) on IM (15) in denotations (14) have the form:

$$\tilde{H} = vx - \frac{q^2x}{2vz^2} + \frac{v^2y^2}{z^2},\ \tilde{V}_1 = x + \frac{vy^2}{z^2},\ \tilde{V}_2 = \frac{vy^2 + xz^2}{2vz^2},$$
$$\tilde{V}_3 = \frac{\left(vy^2 + xz^2\right)\left(-q^2x + \left(G + v^2\right)\left(vy^2 + xz^2\right)\right)}{2z^4},$$

Using above integrals and taking into account expressions for $\lambda_0,\ \lambda_1,\ \lambda_2$ (15), we can write integral K (11) on IMSM (15) as:

$$\tilde{K} = v^2W_{12}\left(W_{21} + 2GW_{12} + 2v^2W_{12}\right)\lambda_3 = v^2W_{12}Q, \tag{17}$$

where $v = x/p,\ W_{12} = (y^2 + pz^2)/2z^2,\ W_{21} = -pq^2/z^2$ are the first integrals of differential equations (16) on IMSM (15). The conditions of stationarity for $\tilde{K}$ (17) enable us to immediately obtain one of stationary solutions of the problem. It has the form

$$v = x/p = (r_1s_1 + r_2s_2)/(r_1^2 + r_2^2) = 0. \tag{18}$$

The rest solutions are determined by equations:

$$2\frac{\partial v}{\partial x_i}W_{12}Q + v(\frac{\partial W_{12}}{\partial x_i}Q + W_{12}\frac{\partial Q}{\partial x_i}) = 0, \quad (i = \overline{1,4}) \tag{19}$$

where $x_1 = r_1,\ x_2 = r_2,\ x_3 = s_1,\ x_4 = s_2$.

We shall not analyze system (19) here, only note that 2nd level IMSM (18) is stable, because v is the first integral of equations (16). We also note that equation (18) defines IM of initial differential equations.

All calculations have been performed with the aid of Mathematica system and program package [7] written in Mathematica language.

The work was supported by the Program of Fundamental Researches of Presidium of the Russian Academy of Sciences no. 17.1.

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