Study of nonlinear degenerated ODEs

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The report describes power transformations of autonomous degenerated ODEs polynomial systems which reduce such systems to a non-degenerate form. There is an example of building exact first integrals of motion of some planar degenerate system in a closed form by the normal form method.

We consider an autonomous degenerated ODEs system of the form

$$\begin{aligned} dx/dt &= -y^3 - b x^3 y + a_0 x^5 + a_1 x^2 y^2 \ , \\ dy/dt &= c x^2 y^2 + x^5 + b_0 x^4 y + b_1 x y^3 \ . \end{aligned} \tag{1}$$

The following result was proven in [4, 5].

Theorem 1. *In the case $D \stackrel{\text{def}}{=} (3b+2c)^2 - 24 \neq 0$, system (1) is locally integrable only if the number $(3b-2c)/\sqrt{D}$ is rational.* When $c = 1/b$ this condition is satisfied. So we put below $c = 1/b$.

Systems with a nilpotent matrix of the linear part were thoroughly studied by Lyapunov and others. In system (1) there is no linear part and the first approximation is not homogeneous. This is the simplest case of a planar system without linear part and with Newton's open polygon [1, 2] consisting of a single edge. In general case such problems have not been studied.

In the report we demonstrate the technique based on the Power Geometry method [3] which allows to transform the problem above to a set of problems with a nilpotent matrix of the linear parts. Really, by using the power transformation [3, 4]

$$x = u v^2, \quad y = u v^3 \tag{2}$$

and the time rescaling $u^2 v^7 dt = d\tau$, we obtain system (1) in the form

$$\begin{aligned} du/d\tau = &\ -3u - [3b + (2/b)]u^2 - 2u^3 + (3a_1 - 2b_1)u^2 v + \\ &\ (3a_0 - 2b_0)u^3 v \ , \\ dv/d\tau = &\ v + [b + (1/b)]uv + u^2 v + (b_1 - a_1)uv^2 + (b_0 - a_0)u^2 v^2 \ . \end{aligned} \tag{3}$$

Under the power transformation (2) the point $x = y = 0$ blows up into two straight invariant lines $u = 0$ and $v = 0$. Along the line $u = 0$ the system (3) has a single stationary point $u = v = 0$. Along the second line $v = 0$ this system has four elementary stationary points

$$u = 0, \quad u = -\frac{1}{b}, \quad u = -\frac{3b}{2}, \quad u = \infty \ . \tag{4}$$

For studying system (1) near the point $x = y = 0$ one needs investigate it near all stationary points (4) of the system (3).

Realization of this approach allowed to get six exact families of the first integrals of motion of (1) in finite terms. Each family is function of two from five parameters of system (1).

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