Recursive technique for evaluation of Feynman diagrams

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Abstract

A method is presented in which matrix elements for some processes are calculated recursively. This recursive calculational technique based on the method of basis spinor, which related with isotropic tetrad in Minkowski space.

Introduction

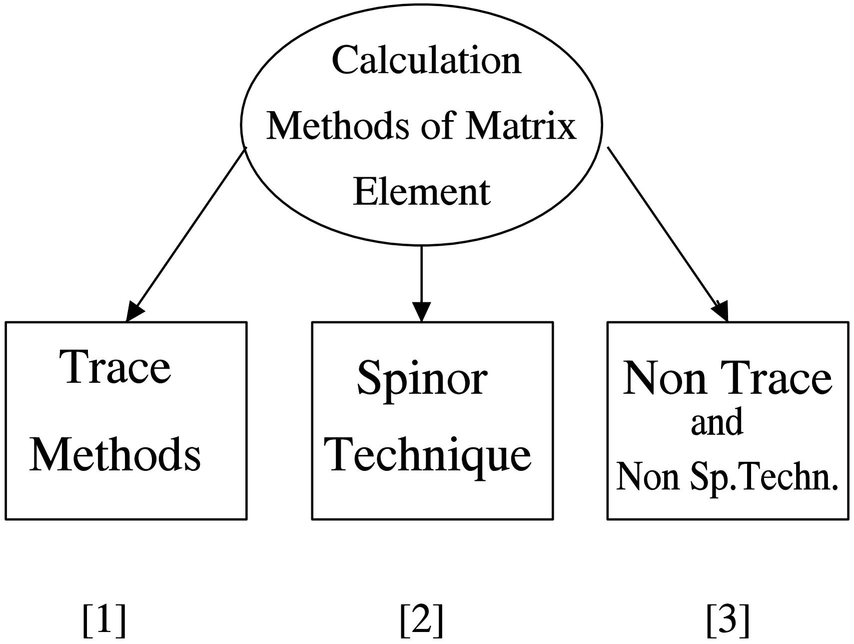

Figure 1: Diagram for the calculation methods of matrix element

The main approach, which has gained popularity in the past decades, is to calculate the Feynman amplitudes directly. Many different methods of calculating the reaction amplitudes with fermions have been developed.

In the general these methods of Feynman amplitudes calculation can be divided in three groups (see Fig.1).

[1]- **Trace methods**:

A.A. Bogush, F.I. Fedorov, Vesti AN BSSR, ser.fiz.-m.n. N 2, 26 (1962).

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F.I. Fedorov, Izvestiya Vyzov. Fizika, N 2, 32 (1980)

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R. Vega and J. Wudka, Phys. Rev. D53, 5286 (1996); *erratum* Phys. Rev. **D56**, 6037 (1997)).

[2]- Spinor Technique:

F.A. Berends, P.H. Davervedt, and R. Kleiss, Nucl. Phys. B253, 441 (1985);

R. Kleiss, W.J. Stirling, Nucl. Phys. B262, 235 (1985);

Zhan Xu, Da-Hua Zhang, and Lee Chang, Nucl.Phys. B291, 392 (1987).

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V.V. Andreev, Phys.Rev. D62, 014029 (2000).

In this method the matrix element is reduced to spinor products of Dirac spinors, i.e. $\bar{u}_{\lambda_p}(p, s_p)\, u_{\lambda_k}(k, s_k)$. The spinor products are calculated through momentum components by means of traces.

[3]- **Nontrace and Non Spinor Techniques**: In these methods the algorithms of reduction of a matrix element to the scalar form differ from the above mentioned methods. ***A. Ballestrero and E. Maina, Phys. Lett. B350, 225 (1995)***
K. Hagiwara, D. Zeppenfeld, Nucl. Phys. B274, 1 (1986)
V.V. Andreev, Nucl.Instr.& Methods in Phys.Research A502 N.2-3, 607 (2003)
It should be noted that there are methods of calculating cross sections without the Feynman diagrams
(see ***F. Caravaglios, M. Moretti, Phys. Lett. B358, 332 (1995)***; ***A. Kanaki and C.G. Papadopoulos, Comput. Phys. Commun. 132, 306 (2000)***.
and calculation methods for some processes (see, ***Carimalo C. , Schiller A. and Serbo V.G. Eur.Phys.J. (2002) V. C23 P.633-649. [e-Print Archive: hep-ph/0112256] and Eur.Phys.J.(2003) V.C29. – P.341-351. [e-Print Archive: hep-ph/0303257].***)

Isotropic tetrad

Let us introduce the orthonormal four-vector basis in Minkowski space which satisfies the relations:

$$l_0^\mu \cdot l_0^\nu - l_1^\mu \cdot l_1^\nu - l_2^\mu \cdot l_2^\nu - l_3^\mu \cdot l_3^\nu = g^{\mu\nu} , \quad (l_A \cdot l_B) = g_{AB} , \tag{1}$$

where g is the Lorentz metric tensor.

With the help of the vectors $l_A \, (A = 0, 1, 2, 3)$ we can define lightlike vectors, which form the isotropic tetrad in Minkowski space

$$\boxed{b_\rho = (l_0 + \rho l_3)/2, n_\lambda = (\lambda \, l_1 + \mathrm{i} l_2)/2 \, , (\rho, \lambda = \pm 1) \, .} \tag{2}$$

From Eqs. (1), (2) it follows that

$$(b_\rho \cdot b_{-\lambda}) = \frac{\delta_{\lambda,\rho}}{2} \; , \quad (n_\lambda \cdot n_{-\rho}) = \frac{\delta_{\lambda,\rho}}{2} \; , \quad (b_\rho \cdot n_\lambda) = 0 \; , \tag{3}$$

$$g^{\mu\nu} = 2 \sum_{\lambda=-1}^{1} \left[b_\lambda^\mu \cdot b_{-\lambda}^\nu + n_\lambda^\mu \cdot n_{-\lambda}^\nu \right] \; . \tag{4}$$

It is always possible to construct the basis of an isotropic tetrad (2) as numerical four-vectors

$$\left(b_{\pm 1}\right)_\mu = (1/2)\left\{1, 0, 0, \pm 1\right\} \; , \; \left(n_{\pm 1}\right)_\mu = (1/2)\left\{0, \pm 1, \mathrm{i}, 0\right\} \tag{5}$$

or by means of physical vectors for reaction.

Also for practical applications it is convenient to introduce the tensor $X^{\mu,\nu}$ which are related to isotropic tetrad vectors (3)

$$X_{A,\,\lambda}^{\mu,\,\nu} = 4\left(b_A^\mu \cdot n_\lambda^\nu - n_\lambda^\mu \cdot b_A^\nu\right) \; , \tag{6}$$

and the contractions with arbitrary four-vectors

$$\begin{aligned} X_{A,\,\lambda}^{p,\,\nu} &\equiv p_\mu X_{A,\,\lambda}^{\mu,\,\nu} \; , \\ X_{A,\,\lambda}^{p,\,q} &\equiv p_\mu \, q_\nu X_{A,\,\lambda}^{\mu,\,\nu} = 4\left((p \cdot b_A)\,(q \cdot n_\lambda) - (p \cdot n_\lambda)\,(q \cdot b_A)\right) \; . \end{aligned} \tag{7}$$

If four-vectors $p = (p^0,\ p^x = p^0 \sin\theta_p \sin\varphi_p,\quad p^y = p^0 \sin\theta_p \cos\varphi_p,$ $p^z = p^0 \cos\theta_p)$ and $q = (q^0,\ q^x = q^0 \sin\theta_q \sin\varphi_q,\quad q^y = q^0 \sin\theta_q \cos\varphi_q,$ $q^z = q^0 \cos\theta_q)$ are lightlike, that the function $X^{p,\,q}_{A,\,\lambda}$ is connected with spinor product of these vectors

$$\langle p\, q\rangle = 2\sqrt{p^0 q^0}\times \\ \times\left[e^{-\mathrm{i}\phi_p}\cos\frac{\theta_p}{2}\sin\frac{\theta_q}{2} - e^{-\mathrm{i}\phi_q}\cos\frac{\theta_q}{2}\sin\frac{\theta_p}{2}\right] \tag{8}$$

by means of

$$X^{p,\,q}_{1,\,1} = 2\sqrt{(p\cdot b_1)\,(q\cdot b_1)}\,\langle p\, q\rangle\ . \tag{9}$$

Polarization Vectors of Massless Bosons

By means of the isotropic tetrad vectors we can determine the polarization vectors of massless (also and massive) vector bosons. For photons with momentum k^μ and helicity $\lambda = \pm 1$ we use following definition of polarizations in the axial gauge

$$\varepsilon_\lambda\,(k) = \frac{-\sqrt{2}\,\lambda\left[(k\cdot n_{-\lambda})\,b_{-1} - (b_{-1}\cdot k)\ \, n_{-\lambda}\right]}{(k\cdot b_{-1})}$$

$$= \frac{-\lambda X^{\mu,\,k}_{-1,-\lambda}}{2\sqrt{2}\,(k\cdot b_{-1})}\,, \qquad (10)$$

provided that, the four-vectors k, b_1, b_{-1} are linearly independent.

Massless basis spinors

With the help of isotropic tetrad

$$\boxed{b_\rho = (l_0 + \rho l_3)/2 \,, n_\lambda = (\lambda l_1 + i l_2)/2 \,, \quad (\rho, \lambda = \pm 1).}$$

we define *basis spinors*

$$\boxed{u_\lambda\,(b_{-1}) \text{ and } u_\lambda\,(b_1)}$$

Definition

$$\not{b}_{-1} u_\lambda\,(b_{-1}) = 0, \quad u_\lambda\,(b_1) \equiv \not{b}_1 u_{-\lambda}\,(b_{-1}) \;\; , \tag{11}$$

$$\omega_\lambda u_\lambda\,(b_{\pm 1}) = u_\lambda\,(b_{\pm 1}) \tag{12}$$

with the matrix $\omega_\lambda = 1/2\,(1 + \lambda\gamma_5)$ and the normalization condition

$$u_\lambda\,(b_{\pm 1})\,\bar{u}_\lambda\,(b_{\pm 1}) = \omega_\lambda\; \not{b}_{\pm 1}. \tag{13}$$

Phase condition. The relative phase between basis spinors with different helicity is given by

$$\not{n}_\lambda u_{-\nu}\left(b_{-1}\right) = \delta_{\lambda,\nu} u_\lambda\left(b_{-1}\right). \tag{14}$$

Some properties of basis spinors

The important property of basis spinors (11) is the ***completeness relation***

$$\sum_{\lambda,A=-1}^{1} u_\lambda\left(b_A\right) \bar{u}_{-\lambda}\left(b_{-A}\right) = I\ , \tag{15}$$

which follows from Eqs.(11),(14). Thus, the arbitrary bispinor can be decomposed in terms of basis spinors $u_\lambda\left(b_A\right)$.

With the help of $g^{\mu\nu} = 2\sum_{\lambda=-1}^{1}\left[b^{\mu}_{\lambda}\cdot b^{\nu}_{-\lambda} + n^{\mu}_{\lambda}\cdot n^{\nu}_{-\lambda}\right]$ Dirac matrix γ^{μ} can be rewritten as

$$\gamma^{\mu} = g_{\mu\nu}\gamma^{\nu} = 2\sum_{\lambda=-1}^{1}\left[\not{b}_{-\lambda} b^{\mu}_{\lambda} + \not{n}_{-\lambda} n^{\mu}_{\lambda}\right] \ . \tag{16}$$

Using the Eqs.(14),(16) we can obtain that

$$\boxed{\gamma^{\mu} u_{\lambda}\left(b_{A}\right) = 2\, b^{\mu}_{A}\, u_{-\lambda}\left(b_{-A}\right) - 2\, A\, n^{\mu}_{-A\times\lambda}\, u_{-\lambda}\left(b_{A}\right) \ ,} \tag{17}$$

$$\boxed{\bar{u}_{\lambda}\left(b_{C}\right) u_{\rho}\left(b_{A}\right) = \delta_{\lambda,-\rho}\, \delta_{C,-A}, \ (C, A, \lambda, \rho) = \pm 1 \ ,} \tag{18}$$

$$\boxed{\omega_{\lambda} u_{\rho}\left(b_{A}\right) = \delta_{\lambda,\rho} u_{\rho}\left(b_{A}\right) \ .} \tag{19}$$

and

$$\boxed{\gamma^{\mu}\gamma^{\nu} u_{\lambda}\left(b_{A}\right) = g^{\mu\nu}\, u_{\lambda}\left(b_{A}\right) - A\, X^{\mu,\nu}_{A,-A\times\lambda}\, u_{\lambda}\left(b_{-A}\right) \ .} \tag{20}$$

Eqs.(17)-(19), and also Eq. (20) underlie the our method .

Recursion relations for matrix element

When evaluating a Feynman amplitude involving the fermions, the amplitude is expressed as sum of terms which have the form

$$\mathcal{M}_{\lambda_p,\lambda_k}(p, s_p,\ k, s_k\ ; Q) = \\ = \mathcal{M}_{\lambda_p,\lambda_k}([p]\,,[k]\,; Q) = \bar{u}_{\lambda_p}(p, s_p)\, Q\ u_{\lambda_k}(k, s_k)\ , \tag{21}$$

where λ_p and λ_k are the polarizations of the external particles with four-momentum p, k and arbitrary polarization vectors s_p, s_k. The operator Q is a sum of products of Dirac γ-matrices. The matrix element (21) with Dirac spinors is a scalar function. As such, it should be expressible in terms of scalar functions formed from the spin and momentum four-vectors of the Dirac spinors, including p, s_p, k, s_k and of the operator Q. We will now that in the our approach this matrix element (21) can be represented as linear combinations of the products of the lower-order matrix element.

Basic matrix element

Let us consider a special case of a matrix element (21), when $p = b_{-C}$ and $k = b_A$, i.e.

$$\begin{aligned} &\mathcal{M}_{-\sigma,\rho}\left(b_{-C},\ b_A\ ; Q\right) \equiv \\ &\equiv \Gamma^{C,A}_{\sigma,\rho}\left[Q\right] \equiv \bar{u}_{-\sigma}\left(b_{-C}\right) Q u_{\rho}\left(b_A\right)\ . \end{aligned} \tag{22}$$

With the help of the completeness relation (15) we can obtain the recursion formula for $\Gamma^{C,A}_{\sigma,\rho}\left[Q_1 Q_2\right]$

$$\Gamma^{C,A}_{\sigma,\rho}\left[Q_1 Q_2\right] = \sum_{D,\lambda=-1}^{1} \Gamma^{C,D}_{\sigma,\lambda}\left[Q_1\right] \Gamma^{D,A}_{\lambda,\rho}\left[Q_2\right]. \tag{23}$$

By means of relations

$$\boxed{\gamma^{\mu} u_{\lambda}\left(b_{A}\right)=2\, b_{A}^{\mu}\, u_{-\lambda}\left(b_{-A}\right)-2\, A\, n_{-A \times \lambda}^{\mu}\, u_{-\lambda}\left(b_{A}\right)\ ,} \tag{24}$$

$$\boxed{\bar{u}_{\lambda}\left(b_{C}\right) u_{\rho}\left(b_{A}\right)=\delta_{\lambda,-\rho}\, \delta_{C,-A},\ (C, A, \lambda, \rho)=\pm 1\ ,} \tag{25}$$

it is easy to calculate $\Gamma_{\sigma,\rho}^{C,A}$ in terms of the isotropic tetrad vectors. For instance,

$$\begin{aligned}
&\Gamma_{\sigma,\rho}^{C,A}\left[\gamma^{\mu}\right]= \\
&=2 \delta_{\sigma,-\rho}\left(\delta_{-C,A}\, b_{A}^{\mu}-A\, \delta_{C,A}\, n_{-A \times \rho}^{\mu}\right)\ , && (26)\\
&\Gamma_{\sigma,\rho}^{C,A}\left[\gamma^{\mu}\, \gamma^{\nu}\right]= \\
&=\delta_{\sigma,\rho}\left(\delta_{C,A} g^{\mu\, \nu}-A\, \delta_{C,-A} X_{A,\,-A \times \rho}^{\mu,\, \nu}\right)\ , && (27)
\end{aligned}$$

where $X_{A,\,-A \times \rho}^{\mu,\, \nu}$ is determined by Eq.(7).

Decomposition coefficients

The next type of lower-order matrix element (21) is

$$\mathcal{M}_{\rho,\lambda_p}(b_A\,,[p]\,;I) \equiv \mathcal{M}_{\rho,\lambda_p}(b_A\,,[p]) = \\ = \bar{u}_\rho(b_A)\,u_{\lambda_p}(p,s_p)\;\; . \qquad (28)$$

The matrix element (28) is determined the decomposition coefficient of an arbitrary Dirac spinor on basis spinors (11).

The opportunity of calculating the coefficients (28) is founded on that an arbitrary Dirac spinor can be determined through the basis spinor (11) with the help of projection operators.

Let us consider massive fermions. The Dirac spinors for massive fermion and antifermion with four-momentum p and arbitrary polarization vector

s_p can be obtained with the help of the relation

$$w^A_\lambda(p, s_p) = \\ = \frac{(\not{p} + A m_p)(1 + \lambda_p \gamma_5 \not{s}_p)}{2\sqrt{(b_{-1}\cdot(p + m_p s_p))}} u_{-A\times\lambda_p}(b_{-1}) \ . \quad (29)$$

The notation $w^A_{\lambda_p}(p, s_p)$ stands for either $u_{\lambda_p}(p, s_p)$ (Dirac spinor of fermion; $A = +1$) or $v_{\lambda_p}(p, s_p)$ (Dirac spinor of antifermion; $A = -1$). The bispinors $u_\lambda(p, s_p)$ and $v_\lambda(p, s_p)$ satisfy Dirac equation and spin conditions for massive fermions and antifermions.

We also found, that the Dirac spinors of fermions and antifermions are related by

$$v_\lambda(p, s_p) = -\lambda\gamma_5\, u_{-\lambda}(p, s_p)\,; \quad \bar{v}_\lambda(p, s_p) = \bar{u}_{-\lambda}(p, s_p)\,\lambda\,\gamma_5\ . \quad (30)$$

The Dirac spinor $u_\lambda(p)$ of massless fermion with momentum p ($p^2 = 0$, $(p\cdot b_{-1}) \neq 0$) and helicity λ is defined by (see, for example, ***R. Kleiss, W.J. Stirling, Nucl. Phys. B262, 235 (1985)***;

$$u_{\lambda}(p) = \frac{\not p}{\sqrt{2\,(p \cdot b_{-1})}} u_{-\lambda}(b_{-1})\,. \tag{31}$$

With help of the Eqs.(29),(31) the matrix element (28) transform to

$$\begin{aligned} &\mathcal{M}_{\rho,\lambda_p}(b_A, [p]) = \\ &= \frac{\bar{u}_{\rho}(b_A)\left[\not\xi^p_1 + \not\xi^p_{-1}\ \not\xi^p_1/m_p\right]\bar{u}_{-\lambda_p}(b_{-1})}{\sqrt{2\,(b_{-1} \cdot \xi^p_1)}} \end{aligned} \tag{32}$$

for the massive fermions with arbitrary vector of polarization and transform to

$$\mathcal{M}_{\rho,\lambda_p}(b_A, p) = \frac{\bar{u}_{\rho}(b_A)\ \not p\ u_{\lambda_p}(b_{-1})}{\sqrt{2\,(b_{-1} \cdot p)}} \tag{33}$$

for massless fermions. Here we have introduced the abbreviations

$$\xi^p_{\pm 1} = \frac{p \pm m_p s_p}{2}\,. \tag{34}$$

Hence the matrix element (28) transform to “fermion string” with massless basis spinors $u_\lambda\left(b_A\right)$.

Using the Eqs.(17)-(19) and (20) the matrix element (28) is reduced to an algebraic expression in terms of scalar products of isotropic tetrad vectors and physical vectors or in terms of components of four-vectors.

Let us consider massless fermions. Using Eqs.(17)-(18) we obtain, that

$$\mathcal{M}_{\rho,\lambda}\left(b_A,p\right)=\delta_{\lambda,-\rho}\sqrt{2}\left[\delta_{A,-1}\sqrt{(p\cdot b_{-1})}-\delta_{A,1}\frac{(p\cdot n_{-\lambda})}{\sqrt{(p\cdot b_{-1})}}\right]. \tag{35}$$

For numerical calculations, as well as in the case of spinor techniques, it is convenient to determine the (35) through the momentum components $p=(p^0,\ \ p^x=p^0\sin\theta_p\sin\varphi_p,\ p^y=p^0\sin\theta_p\cos\varphi_p,\ p^z=p^0\cos\theta_p)$

$$\begin{aligned}\mathcal{M}_{\rho,\lambda}\left(b_A,p\right)&=\delta_{\lambda,-\rho}\left[\delta_{A,-1}\sqrt{p^+}-\delta_{A,1}\lambda\exp\left(\mathrm{i}\lambda\varphi_p\right)\sqrt{p^-}\right]\\&=\delta_{\lambda,-\rho}\sqrt{2p_0}\left[\delta_{A,-1}\cos\frac{\theta_p}{2}-\delta_{A,1}\lambda\sin\frac{\theta_p}{2}\exp\left(\mathrm{i}\lambda\varphi_p\right)\right],\end{aligned} \tag{36}$$

where

$$p^{\pm} = p^0 \pm p^z, \quad p^x + \mathrm{i}\lambda p^y = \sqrt{\left(p^x\right)^2 + \left(p^y\right)^2} \exp\left(\mathrm{i}\lambda\varphi_p\right).$$

Let us consider massive Dirac massive particles with arbitrary polarization vector s_p. After evaluations we obtain, that the decomposition coefficients for a massive fermion with momentum p, an arbitrary polarization vector s_p and mass m_p can be written as scalar products of tetrad and physical vectors

$$\mathcal{M}_{\rho,\lambda_p}\left(b_A, p, s_p\right) = \frac{1}{\sqrt{2\left(b_{-1}\cdot\xi_1^p\right)}}$$
$$\left[2\delta_{\lambda,-\rho}\left\{\delta_{A,-1}\left(b_{-1}\cdot\xi_1^p\right) + \delta_{A,1}\left(n_{-\lambda_p}\cdot\xi_1^p\right)\right\} + \right.$$
$$\left. +\delta_{\lambda,\rho}\left\{\delta_{A,1}\frac{m_p}{2} + \delta_{A,-1}\frac{X_{-1,-\lambda_p}^{\xi_{-1}^p,\xi_1^p}}{m_p}\right\}\right], \tag{37}$$

where where $X^{p,q}$ is determined by Eq.(7).

If the polarization vector s_p of fermion is defined as follows (so called $BDKS$ polarization states)

F.A. Berends, P.H. Davervedt, and R. Kleiss, Nucl. Phys. B253, 441 (1985);

R. Kleiss, W.J. Stirling, Nucl. Phys. B262, 235 (1985);

A. Ballestrero and E. Maina, Phys. Lett. B350, 225 (1995)

$$s_p \equiv s_{KS} = \frac{p}{m_p} - m_p \frac{b_{-1}}{(p \cdot b_{-1})} \; , \tag{38}$$

that the matrix element (37) have a compact form

$$\mathcal{M}_{\rho,\lambda}\left(b_A, p, s_{KS}\right) = \sqrt{2}\delta_{\lambda,-\rho}\left[\delta_{A,-1}\sqrt{(p \cdot b_{-1})} + \delta_{A,1}\frac{(p \cdot n_\lambda)}{\sqrt{(p \cdot b_{-1})}}\right] + \\ +\delta_{\lambda,\rho}\,\delta_{A,1}\frac{m_p}{\sqrt{2\,(p \cdot b_{-1})}} \; . \tag{39}$$

The matrix element (28) with the antifermion can be easily obtained with the help of Eq.(30):

$$\widetilde{\mathcal{M}}_{\rho,\lambda_p}(b_A\ ,[p]) = \bar{u}_\rho(b_A)\, v_{\lambda_p}(p,s_p) =$$
$$= \rho\lambda_p \mathcal{M}_{\rho,-\lambda_p}(b_A\ ,[p])\ . \tag{40}$$

Recursion relation

With the help of completeness relation (15) the amplitude (21) with $Q = Q_2 Q_1$ is expressed as combinations of the lower-order matrix element

$$\mathcal{M}_{\lambda_p,\lambda_k}\left([p]\,,[k]\,;Q_2Q_1\right) =$$
$$= \sum_{\sigma,A=-1}^{1} \mathcal{M}_{\lambda_p,\sigma}\left([p]\,,b_A;Q_2\right)\mathcal{M}_{-\sigma,\lambda_k}\left(b_{-A},[k]\,;Q_1\right)\;. \qquad (41)$$

This insertion allows us to "cut" fermion chain (21) into pieces of fermion chains with basis spinors $u_\lambda\left(b_A\right)$ (see Fig.2)

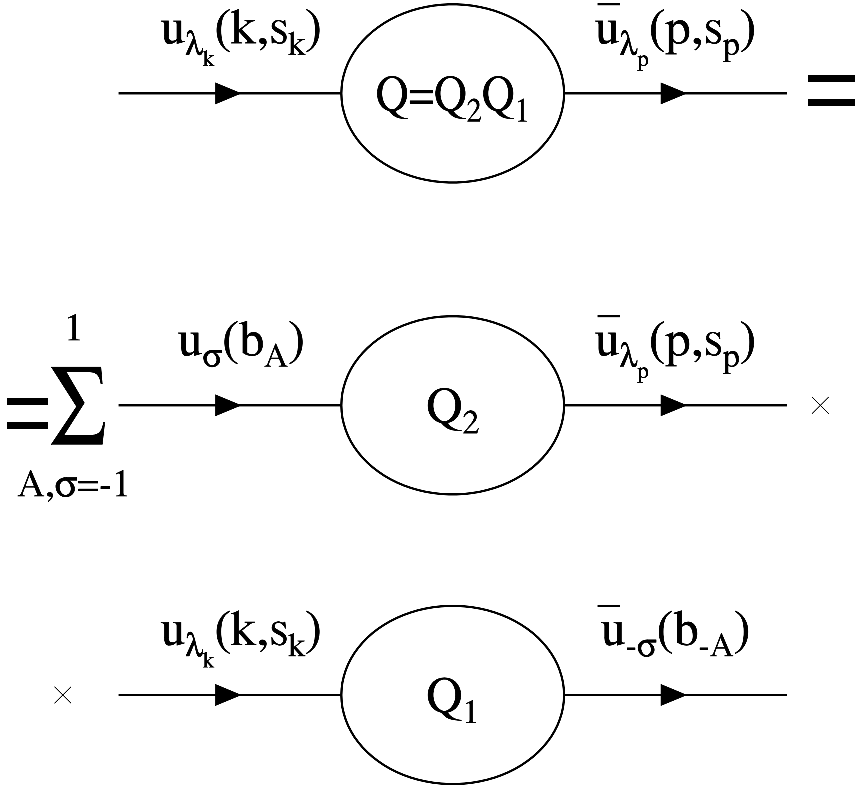

Figure 2: Diagram for the recursion relation (41)

Hence the our formalism enables to calculate the blocks of the Feynman diagrams and then to use them in the calculation as scalar functions. All possible Feynman amplitudes can be built up from a set of “building” blocks.

Let us consider the matrix element (21) with an operator

$$Z^{(n)} = Q_n Q_{n-1} \cdots Q_1 Q_0 \tag{42}$$

with $Q_0 = I$. In the Eq.(42) all operators Q_j have an identical mathematical expressions. Using Eq.(41) we find that

$$\mathcal{M}_{\lambda_p,\lambda_k}\left([p]\,,[k]\,;Z^{(n)}\right) \equiv \mathcal{M}^{(n)}_{\lambda_p,\lambda_k}\left([p]\,,[k]\right) =$$
$$= \sum_{\sigma,A=-1}^{1} \mathcal{M}_{\lambda_p,\sigma}\left([p]\,,b_A\right)\mathcal{M}^{(n)}_{-\sigma,\lambda_k}\left(b_{-A},[k]\right)\ , \tag{43}$$

where matrix element $\mathcal{M}^{(n)}_{-\sigma,\lambda_k}\left(b_{-A},[k]\right)$ can be calculated with the help of recursion relation

$$\mathcal{M}^{(n)}_{-\sigma,\lambda_k}\left(b_{-A},[k]\right) =$$
$$= \sum_{\rho,C=-1}^{1} \Gamma^{A,C}_{\sigma,\rho}\left[Q_n\right]\mathcal{M}^{(n-1)}_{-\rho,\lambda_k}\left(b_{-C},[k]\right)\ . \tag{44}$$

Examples

"Toy" example

$$\mathcal{M}^{(n)}_{\lambda_p,\lambda_k}(p,s_p,k,s_k) = \\ = \bar{u}_{\lambda_p}(p,s_p)\, \not{q}_n\, \not{q}_{n-1} \ldots \not{q}_1 u_{\lambda_k}(k,s_k)\,, \qquad (45)$$

where q_j are some arbitrary four-vectors.

Therefore, we have that

$$Z^{(n)} = \not{q}_n\, \not{q}_{n-1} \ldots \not{q}_1\,. \qquad (46)$$

With the help of Eqs.(43) and (44) we can obtain the recursion formulas for calculation matrix element (45):

$$\mathcal{M}^{(j)}_{\lambda_p,\lambda_k}\left([p]\,,[k]\right) =$$

$$= 2\sum_{\rho,C=-1}^{1}\mathcal{M}_{\lambda_p,\rho}\left([p]\,,b_C\right)\Big[\left(q_j\cdot b_{-C}\right)\mathcal{M}^{(j-1)}_{\rho,\lambda_k}\left(b_C,[k]\right) -$$

$$-C\ \left(q_j\cdot n_{C\times\rho}\right)\mathcal{M}^{(j-1)}_{\rho,\lambda_k}\left(b_{-C},[k]\right)\Big]\ , \tag{47}$$

and

$$\mathcal{M}^{(j)}_{\lambda_p,\lambda_k}\left([p]\,,[k]\right) =$$

$$= \sum_{\rho,C=-1}^{1}\mathcal{M}_{\lambda_p,\rho}\left([p]\,,b_C\right)\Big[\left(q_j\cdot q_{j-1}\right)\mathcal{M}^{(j-2)}_{-\rho,\lambda_k}\left(b_{-C},[k]\right) +$$

$$+C\ X^{q_j,\,q_{j-1}}_{-C,C\times\rho}\mathcal{M}^{(j-2)}_{-\rho,\lambda_k}\left(b_C,[k]\right)\Big]\ . \tag{48}$$

Photon emission

Let us consider the matrix element where photon emission occurs from the incoming electron

$$\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2],[p_1];Q\ Z_k\right) = \bar{u}_{\lambda_2}\left(p_2, s_{p_2}\right) Q\ Z_k\ u_{\lambda_1}\left(p_1, s_{p_1}\right) \quad \text{, where} \tag{49}$$

$$Z_k = \frac{\left(\not{p}_1 - \not{k} + m_e\right)\ \not{\varepsilon}_\sigma\left(k\right)}{\left(p_1 - k\right)^2 - m_e^2}\ . \tag{50}$$

Using the algebra of γ-matrix and Dirac equation the operator Z_k rewritten as

$$Z_k = \frac{\left(p_1\ \varepsilon_\sigma\left(k\right)\right)}{\left(p_1\ k\right)} - \frac{\not{k}\ \not{\varepsilon}_\sigma\left(k\right)}{2\left(p_1\ k\right)}\ . \tag{51}$$

Now we get

$$\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2],[p_1];Q\ Z_k\right)=\frac{(p_1\ \varepsilon_\sigma(k))}{(p_1\ k)}\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2],[p_1];Q\right)-$$
$$-\frac{1}{2\,(p_1\ k)}\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2],[p_1];Q\ \not{k}\ \not{\varepsilon}_\sigma(k)\right)\ . \tag{52}$$

The recursion technique (see Eq.(41)) imply

$$\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2],[p_1];Q\ \not{k}\ \not{\varepsilon}_\sigma(k)\right)=$$
$$=\sum_{\sigma,C=-1}^{1}\mathcal{M}_{\lambda_2,\sigma}\left([p_2],b_C;Q\right)\mathcal{M}_{-\sigma,\lambda_1}\left(b_{-C},[p_1];\not{k}\ \not{\varepsilon}_\sigma(k)\right)$$
$$=\sum_{\sigma,C=-1}^{1}\mathcal{M}_{\lambda_2,\sigma}\left([p_2],b_C;Q\right)\mathcal{M}_{-\sigma,\lambda_1}\left(b_C,[p_1]\right)C\ X^{k,\varepsilon_\sigma(k)}_{-C,C\times\sigma}\ . \tag{53}$$

By means of the Eqs.(7),(10) we have that

$$C\ X^{k,\varepsilon_\sigma(k)}_{-C,C\times\sigma}=\sqrt{2}(-C)\ \sigma\left(k\ b_{-C}\right)e^{-\mathrm{i}(1-C)\varphi_k} \tag{54}$$

For the matrix element (49) with the fermions and photon of arbitrary polarization we have the exact formula in terms of lower-order matrix elements $\mathcal{M}_{\lambda_2,\sigma}\left([p_2]\,, b_C; Q\right)$ and $\mathcal{M}_{-\sigma,\lambda_1}\left(b_C, [p_1]\right)$

$$\begin{aligned}
&\mathcal{M}_{\lambda_2,\lambda_1}\left([p_2]\,, [p_1]\,; Q \;\; \not{k}\; \not{\varepsilon}_\sigma\left(k\right)\right) = \\
&= \frac{\left(p_1\;\varepsilon_\sigma\left(k\right)\right)}{\left(p_1\;k\right)} \mathcal{M}_{\lambda_2,\lambda_1}\left([p_2]\,, [p_1]\,; Q\right) + \frac{1}{\sqrt{2}\left(p_1\;k\right)} \times \\
&\times \sum_{\sigma,C=-1}^{1} \mathcal{M}_{\lambda_2,\sigma}\left([p_2]\,, b_C; Q\right) \\
&\left\{\mathcal{M}_{-\sigma,\lambda_1}\left(b_C, [p_1]\right) C\;\sigma\left(k\;b_{-C}\right) e^{-\mathrm{i}(1-C)\varphi_k}\right\}\;, \qquad (55)
\end{aligned}$$

where $\mathcal{M}_{-\sigma,\lambda_1}\left(b_C, [p_1]\right)$ is determined by the Eq.(37) for arbitrary fermion polarization or Eqs.(36), (39) for special cases of fermion polarization.

The process $e^+e^- \to n\gamma$

Consider the process

$$e^+ (p_2, \sigma_2) + e^- (p_1, \sigma_1) \to \gamma (k_1, \lambda_1) + \gamma (k_2, \lambda_2) + \cdots + \gamma (k_n, \lambda_n) \,, \qquad (56)$$

where the momenta of the particles and spin numbers are given between parentheses.

The Feynman diagrams of the processes (56) contain the matrix element

$$\begin{aligned} & M^{(\lambda_1,\lambda_2,\ldots\lambda_n)}_{\sigma_2,\sigma_1} (p_2, s_{p_2}, p_1, s_{p_1}; k_1, k_2, \ldots k_n) = \\ & = \mathcal{M}^{(n)}_{\sigma_2,\sigma_1} ([p_2], [p_1]) = \bar{v}_{\sigma_2} (p_2, s_{p_2}) \not{\epsilon}_{\lambda_n} (k_n) \cdots \\ & \cdots \not{\epsilon}_{\lambda_3} (k_3) \frac{\not{Q}_2 + m}{Q_2^2 - m^2} \not{\epsilon}_{\lambda_2} (k_2) \frac{\not{Q}_1 + m}{Q_1^2 - m^2} \not{\epsilon}_{\lambda_1} (k_1) \, u_{\sigma_1} (p_1, s_{p_1}) + \\ & + (n! - 1) \, \text{other permutations of} \, (1, 2, \ldots, n) \ , \end{aligned} \qquad (57)$$

where $Q_j = p_1 - \sum_{i=1}^{j} k_i$. Hence, we have that

$$\mathcal{M}^{(n)}_{\sigma_2,\sigma_1}\left([p_2],[p_1]\right) = \sum_{\rho,C=-1}^{1} \widetilde{\mathcal{M}}_{\sigma_2,\rho}\left([p_2],b_C;\not{\epsilon}_{\lambda_n}(k_n)\right)\mathcal{M}^{(n-1)}_{-\rho,\sigma_1}\left(b_{-C},[p_1]\right) \quad (58)$$

with

$$\widetilde{\mathcal{M}}_{\sigma_2,\rho}\left([p_2],b_C;\not{\epsilon}_{\lambda_n}(k_n)\right) = \bar{v}_{\sigma_2}\left(p_2,s_{p_2}\right)\not{\epsilon}_{\lambda_n}(k_n)\,u_\rho\left(b_C\right) \ . \quad (59)$$

Using the expressions (27) and (10) we obtain, that (59) is determined by

$$\begin{aligned}&\widetilde{\mathcal{M}}_{\sigma_2,\rho}\left([p_2],b_C;\not{\epsilon}_{\lambda_n}(k_n)\right) = \\ &= -\sqrt{2}\,\rho\,\sigma_2\left[\delta_{C,1}\cot\frac{\theta_n}{2}\,e^{\mathrm{i}\lambda_n\varphi_n}\mathcal{M}_{-\sigma_2,-\rho}\left([p_2],b_{-1}\right)+\right. \\ &\left.+\rho\delta_{C,-\rho\times\lambda_n}\mathcal{M}_{-\sigma_2,-\rho}\left([p_2],b_{-\rho\times\lambda_n}\right)\right] \quad (60)\end{aligned}$$

for the $k_n = k^0\left(1,\sin\theta_n\cos\varphi_n,\sin\theta_n\sin\varphi_n,\cos\theta_n\right)$.

The final recursion relation of the process (56) with arbitrary polarization states of the photons and fermions is written as

$$\mathcal{M}^{(n)}_{\sigma_2,\sigma_1}\left([p_2]\,,[p_1]\right) = \sum_{\rho=-1}^{1} -\sqrt{2}\,\rho\,\sigma_2 \times$$

$$\times\left[\cot\frac{\theta_n}{2}\,e^{\mathrm{i}\lambda_n\varphi_n}\mathcal{M}_{-\sigma_2,-\rho}\left([p_2]\,,b_{-1}\right)\mathcal{M}^{(n-1)}_{-\rho,\sigma_1}\left(b_{-1},[p_1]\right) + \right.$$

$$\left. +\rho\mathcal{M}_{-\sigma_2,-\rho}\left([p_2]\,,b_{-\rho\times\lambda_n}\right)\mathcal{M}^{(n-1)}_{-\rho,\sigma_1}\left(b_{\rho\times\lambda_n},[p_1]\right)\right]\;, \tag{61}$$

where (see Eq.(47) and Eq.(48))

$$\mathcal{M}^{(j)}_{-\rho,\sigma_1}\left(b_{-C},[k]\right) = \frac{1}{Q_j^2-m^2}\,\{$$

$$2m\,[\,(\varepsilon_j\cdot b_{-C})\,\mathcal{M}^{(j-1)}_{\rho,\sigma_1}\left(b_C,[k]\right) - C\;\left(\varepsilon_j\cdot n_{C\rho}\right)\mathcal{M}^{(j-1)}_{\rho,\sigma_1}\left(b_{-C},[k]\right)\,] +$$

$$+\left(Q_j\cdot\varepsilon_j\right)\mathcal{M}^{(j-1)}_{-\rho,\sigma_1}\left(b_{-C},[k]\right) + C\;X^{Q_j,\,\varepsilon_j}_{-C,C\times\rho}\mathcal{M}^{(j-1)}_{-\rho,\sigma_1}\left(b_C,[k]\right)\,\}\;, \tag{62}$$

with $\varepsilon_j = \varepsilon_{\lambda_j}\left(k_j\right)$ and the scalar function $X^{p,q}_{A,\rho}$.

Summary

In present paper we have formulated a new effective method to calculate the Feynman amplitudes for various processes with fermions of arbitrary polarizations. In our method it is much easier to keep track of partial results and to set up recursive schemes of evaluation which compute and store for later use subdiagrams of increasing size and complexity. In our approach of the matrix element calculation:

1 We don't use an explicit form of bispinors and γ-matrices

2 We don't use or operation of trace calculations

3 In this method as well as in the trace methods the matrix element of Feynman amplitudes is reduced to the combination of scalar products of momenta and polarization vectors.

4 Unlike spinor technique in different variants
(***F.A. Berends, P.H. Davervedt, and R. Kleiss, Nucl. Phys.***

B253, 441 (1985); ***R. Kleiss, W.J. Stirling, Nucl. Phys. B262, 235 (1985)***; ***Zhan Xu, Da-Hua Zhang, and Lee Chang, Nucl.Phys. B291, 392 (1987)*** and others)
this method doesn't use either Chisholm identities, or the presentation of the contraction $\not p$ with four vector p and of the polarization vector of bosons through the bispinors.

5 Unlike `Wvd` technique ***F.A. Berends and W.T. Giele, Nucl. Phys. B294, 700 (1987),***
S. Dittmaier, Phys. Rev. D59, ***016007 (1999)***
in this method doesn't use special Feynman rules for calculating of the matrix elements.

The recursive algorithms can be easily realized in the various systems of symbolic calculation (`Mathematica, Maple, Reduce, Form`) and in such packages as `FeynArts`
J. Küblbeck, M. Böhm, and A. Denner, Comp. Phys. Commun.

60, 165 (1990),

`FeynCalc`

R. Mertig, M.Böhm, and A. Denner, Comp. Phys. Commun. 64, 345 (1991),

`HIP`

E.Yehudai, Preprint Fermi-Lab-Pub-92/22-T (1992) and so on.

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