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Conservation laws and Hamilton's equations for systems with long-range interaction and memory

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Abstract

Using the fact that extremum of variation of generalized action can lead to the fractional dynamics in the case of systems with long-range interaction and long-term memory function, we consider two different applications of the action principle: generalized Noether's theorem and Hamiltonian type equations. In the first case, we derive conservation laws in the form of continuity equations that consist of fractional time–space derivatives. Among applications of these results, we consider a chain of coupled oscillators with a power-wise memory function and power-wise interaction between oscillators. In the second case, we consider an example of fractional differential action 1-form and find the corresponding Hamiltonian type equations from the closed condition of the form.

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1. Introduction

Different physical phenomena such as anomalous transport or random walk with infinite moments [1,2], dynamics of porous media [3,4], continuous time random walk [5–7], chaotic dynamics [8] (see also reviews [9,10]) can be described by equations with fractional integro-differentiation. Despite of fairly deep and comprehensive results in fractional calculus (see [11–14]) a possibility of their applications to physics needs to develop specific physical tools such as extension of fractional calculus to the areas as multi-dimension [11,17], multi-scaling [15,16], variational principles [18,19].

In this paper, we concentrate on two problems important for numerous physical applications: conservation laws and Hamiltonian type equations, both obtained from the corresponding fractional action principles.

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In Section 2, we derive the Noether’s theorem for a Lagrangian that includes non-local space–time densities. The Noether’s theorem was also discussed in [23,24]. Our new derivation shows in an explicit way how fractional derivative in time emerges from the specific type of the memory function, and how fractional derivative in space is related to a specific long-distance potential of interaction (Section 3.) In Section 4, these results are applied to a chain of nonlinear oscillators that is a subject of great interest in statistics and dynamics [25,26]. Finally, at Section 5, we derive a specific case of fractional Hamilton’s equations. Different steps in this direction were performed in [29–31]. We consider the Lagrangian density as a functional without fractional derivatives but, instead, the differential 1-form has fractional differentials. Some examples are given for this type of systems.

The main feature of this paper is the consideration of fractional type differentials or derivatives in both space–time coordinates.

2. Noether’s theorem for long-range interaction and memory

2.1. Action and Lagrangian functionals

Let us consider the action functional

$$S[u] = \int_R d^2x \int_R d^2y \mathcal{L}(u(x), u(y), \partial u(x), \partial u(y)), \tag{1}$$

where $x = (t, r)$, t is time, r is coordinate, and $y = (t', r')$, $\partial u(x) = (\partial_t u(t, r), \partial_r u(t, r))$. The integration is carried out over a region R of the two-dimensional space \mathbb{R}^2 to which x belong. The field $u(x)$ is defined in the region R of \mathbb{R}^2 . We assume that $u(x)$ has partial derivatives

$$\partial_0 u(x) = \frac{\partial u(t, r)}{\partial t}, \quad \partial_1 u(x) = \frac{\partial u(t, r)}{\partial r},$$

which are smooth functions with respect to time and coordinate. Here $\mathcal{L}(u(x), u(y), \partial u(x), \partial u(y))$ is generalized density of Lagrangian. If

$$\mathcal{L}(u(x), u(y), \partial u(x), \partial u(y)) = \mathcal{L}(u(x), \partial u(x))\delta(x - y), \tag{2}$$

then we have the usual action functional

$$S[u] = \int_R d^2x \mathcal{L}(u(x), \partial u(x)).$$

The variation of the action (1) is

$$\delta S[u, h] = \int_R d^2x \int_R d^2y \left(\frac{\partial \mathcal{L}}{\partial u(x)} h(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u(x))} \partial_\mu h(x) + \frac{\partial \mathcal{L}}{\partial u(y)} h(y) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u(y))} \partial_\mu h(y) \right), \tag{3}$$

where $\mu = 0, 1$, $\partial_\mu = \partial/\partial x^\mu$ and

$$\mathcal{L} = \mathcal{L}(u(x), u(y), \partial u(x), \partial u(y))$$

and $h(x) = \delta u(x)$ is the variation of the field u . The variation (3) can be presented as

$$\delta S[u, h] = \int_R d^2x \int_R d^2y \left(\frac{\partial \mathcal{L}_s}{\partial u(x)} h(x) + \frac{\partial \mathcal{L}_s}{\partial(\partial_\mu u(x))} \partial_\mu h(x) \right), \tag{4}$$

where

$$\mathcal{L}_s = \mathcal{L}(u(x), u(y), \partial u(x), \partial u(y)) + \mathcal{L}(u(y), u(x), \partial u(y), \partial u(x)).$$

We can define the functional

$$L[x, u, \partial u] = \frac{1}{2} \int_R d^2y \mathcal{L}_s, \tag{5}$$

which will be called the Lagrangian functional. For (2), the functional (5) is equal to the usual Lagrangian density $\mathcal{L}(u, \partial u)$.

Using notations

$$\frac{\delta L[x, u, \partial u]}{\delta u(x)} = \int_R d^2y \frac{\partial \mathcal{L}_s}{\partial u(x)}, \quad \frac{\delta L[x, u, \partial u]}{\delta(\partial_\mu u(x))} = \int_R d^2y \frac{\partial \mathcal{L}_s}{\partial(\partial_\mu u(x))}, \tag{6}$$

the variation (4) is

$$\delta S[u, h] = \int_R d^2x \left(\frac{\delta L[x, u, \partial u]}{\delta u(x)} h(x) + \frac{\delta L[x, u, \partial u]}{\delta(\partial_\mu u(x))} \partial_\mu h(x) \right). \tag{7}$$

The structure of (7) is non-local, i.e., it includes, through (5) and \mathcal{L}_s , a possibility of the long-term memory and long-range interaction. Let us derive the generalization of the Noether’s theorem for this case in the way analogical to the local case (2).

2.2. Equations of motion

First, we separate the variation that is linked in the variation of coordinates

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu, \tag{8}$$

and the variation caused by a change of the form of u ,

$$u(x) \rightarrow u'(x) = u(x) + \delta u(x).$$

The variation $h(x) = \delta u(x)$ of $u(x)$ that is not due to the variation in coordinates is called local. The total variation is

$$\Delta u(x) = u'(x') - u(x) = \delta u(x) + (\partial u / \partial x^\mu) \delta x^\mu \tag{9}$$

in the first approximation with respect to δx . Let us consider the variation of (1) as

$$\delta S[u, h] = \int_R d^2x' L[x', u', \partial' u'] - \int_R d^2x L[x, u, \partial u]. \tag{10}$$

The elements of the two-dimensional volume in new and old coordinates are related through the formula

$$d^2x' = J(x'/x) d^2x,$$

where $J(x'/x) = \det|\partial x'^\mu / \partial x^\nu|$ is the Jacobian of the transformation. Using the well-known relation

$$\det A = \exp \text{Tr} \ln A, \tag{11}$$

and the linear approximation

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu^\mu + \partial_\nu(\delta x^\mu),$$

we get

$$J(x'/x) = 1 + \partial_\mu(\delta x^\mu).$$

Here δ_ν^μ is the Kronecker symbol.

For the variation (10), we get

$$\delta S[u, h] = \int_R d^2x (\delta L + L \partial_\mu(\delta x^\mu)) = \int_R d^2x \left(\frac{\delta L}{\delta u} \delta u + \frac{\delta L}{\delta(\partial_\mu u)} \delta(\partial_\mu u) + \partial_\mu(L \delta x^\mu) \right), \tag{12}$$

where the variations of L are defined in (6). Using $\delta(\partial_\mu u) = \partial_\mu(\delta u)$, and

$$\frac{\delta L}{\delta(\partial_\mu u)} \partial_\mu \delta u = \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu u)} \delta u \right) - \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu u)} \right) \delta u,$$

we can rewrite (12) as

$$\delta S[u, h] = \int_R d^2x \left(\frac{\delta L}{\delta u} - \partial_\mu \frac{\delta L}{\delta(\partial_\mu u)} \right) \delta u + \int_R d^2x \partial_\mu \left(L \delta x^\mu + \frac{\delta L}{\delta(\partial_\mu u)} \delta u \right). \tag{13}$$

The Gauss theorem gives

$$\int_R d^2x \partial_\mu \left(L \delta x^\mu + \frac{\delta L}{\delta(\partial_\mu u)} \delta u \right) = \int_{\partial R} dS_\mu \left(L \delta x^\mu + \frac{\delta L}{\delta(\partial_\mu u)} \delta u \right). \tag{14}$$

We assume that at the boundary of the domain of integration the function $u(x)$ is selected in a definite manner (the boundary condition). Then stationary (a minimum or a saddle point) values of $S[u]$ from variational equations $\delta S[u, h] = 0$ with

$$[\delta u]_{\partial R} = 0, \quad [\delta x]_{\partial R} = 0$$

at the boundary ∂R of the domain of integration, constitutes the necessary and sufficient condition for the real evolution of the field, that is, that $u = u(x)$ represents the true dynamics under the given boundary conditions. The stationary action principle gives

$$\frac{\delta L[x, u, \partial u]}{\delta u(x)} - \partial_\mu \frac{\delta L[x, u, \partial u]}{\delta(\partial_\mu u(x))} = 0, \tag{15}$$

where the variations of L are defined in (6). This equation is the Euler–Lagrange equation for Lagrangian functional $L[x, u, \partial u]$.

Let us consider three special cases of Eq. (15).

- (1) The absence of the memory and long-range interaction means that

$$\mathcal{L}(u(x), u(y), \partial u(x), \partial u(y)) = \mathcal{L}(u(x), \partial u(x)) \delta(x - y).$$

Then (15) gives the usual Euler–Lagrange equation

$$\frac{\partial \mathcal{L}(u, \partial u)}{\partial u(x)} - \partial_\mu \frac{\partial \mathcal{L}(u, \partial u)}{\partial(\partial_\mu u(x))} = 0. \tag{16}$$

- (2) If the generalized Lagrangian density is

$$\mathcal{L}(u(x), u(y), \partial u(x), \partial u(y)) = \mathcal{L}(u(x), \partial u(x)) c_1(D, r) \delta(x - y),$$

where

$$c_1(D, r) = \frac{|r|^{D-1}}{\Gamma(D)} \quad (0 < D < 1)$$

for a medium distributed on \mathbb{R}^1 with the fractional Hausdorff dimension D , then Eq. (15) has the form

$$c_1(D, r) \frac{\partial \mathcal{L}(u, \partial u)}{\partial u(x)} - \partial_\mu \left(c_1(D, r) \frac{\partial \mathcal{L}(u, \partial u)}{\partial(\partial_\mu u(x))} \right) = 0. \tag{17}$$

This is Euler–Lagrange equation for the field $u(x) = u(t, r)$ in fractal medium. Examples of the field (wave) equations for fractal medium string and fractional hydrodynamics are considered in [33]. For example,

$$\mathcal{L}(u(x), \partial u(x)) = \frac{1}{2} (\partial_t u(t, r))^2 - \frac{1}{2} v^2 (\partial_r u(t, r))^2,$$

leads to the equation

$$c_1(D, r) \partial_t^2 u(t, r) - v^2 \partial_r (c_1(D, r) \partial_r u(t, r)) = 0,$$

that describes the propagation waves in fractal medium.

(3) Consider the action functional

$$S[u] = \int_R d^2x \int_R d^2y \left(\frac{1}{2} \partial_t u(x) g_0(x, y) \partial_r u(y) - \frac{1}{2} \partial_r u(x) g_1(x, y) \partial_t u(y) - V(u(x), u(y)) \right), \tag{18}$$

where the kernels $g_0(x, y)$ and $g_1(x, y)$ are responsible for non-local time-coordinates dynamics and V is non-local interaction potential. Then the Lagrangian functional (5) is

$$L[x, u, \partial_t u, \partial_r u] = \int_R d^2y \left(\frac{1}{2} \partial_t u(x) K_0(x, y) \partial_r u(y) - \frac{1}{2} \partial_r u(x) K_1(x, y) \partial_t u(y) - U(u(x), u(y)) \right), \tag{19}$$

where

$$\begin{aligned} K_0(x, y) &= \frac{1}{2} [g_0(x, y) + g_0(y, x)], \\ K_1(x, y) &= \frac{1}{2} [g_1(x, y) + g_1(y, x)], \\ U(u(x), u(y)) &= V(u(x), u(y)) + V(u(y), u(x)). \end{aligned} \tag{20}$$

Assume that

$$U(u(x), u(y)) = U(u(x)) \delta(x - y).$$

In this case, the Euler–Lagrange functional Eq. (15) has the form (see also [22])

$$\int_R d^2y \partial_t K_0(x, y) \partial_r u(y) - \int_R d^2y \partial_r K_1(x, y) \partial_t u(y) + \frac{\partial U(u(x))}{\partial u(x)} = 0. \tag{21}$$

It is an integro-differential equation, which allows us to derive field equations for different cases of the kernels $K_0(x, y)$ and $K_1(x, y)$.

In the absence of memory and for local interaction the kernels (20) are defined at the only instant t and point r , i.e.,

$$K_0(x, y) = g_0 \delta(x - y), \quad K_1(x, y) = g_1 \delta(x - y)$$

with some constants g_0 and g_1 . Then Eq. (21) gives

$$g_0 \partial_t^2 u(t, r) - g_1 \partial_r^2 u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0.$$

For example $g_0 = g_1 = 1$, when

$$U(u(t, r)) = -\cos u(t, r),$$

we get the sine-Gordon equation

$$\partial_t^2 u(t, r) - \partial_r^2 u(t, r) + \sin u(t, r) = 0. \tag{22}$$

2.3. Noether’s current

Let us derive the Noether’s current by using the action variation (13).

The second integral of (13) can be presented as

$$\int_R d^2x \partial_\mu \left(L \delta x^\mu + \frac{\delta L}{\delta(\partial_\mu u)} \delta u \right) = \int_R d^2x \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu u)} [\delta u + (\partial_\nu u) \delta x^\nu] - \left[\frac{\delta L}{\delta(\partial_\mu u)} (\partial_\nu u) - \delta_\nu^\mu L \right] \delta x^\nu \right). \tag{23}$$

Using the total variation (9), Eq. (23), and the energy–momentum tensor

$$\theta_\nu^\mu = \frac{\delta L}{\delta(\partial_\mu u)} (\partial_\nu u) - \delta_\nu^\mu L, \tag{24}$$

the variation (13) yields

$$\delta S[u, h] = \int_R d^2x \left(\frac{\delta L}{\delta u} - \partial_\mu \frac{\delta L}{\delta(\partial_\mu u)} \right) \delta u + \int_R d^2x \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu u)} \Delta u - \theta_\nu^\mu \delta x^\nu \right). \tag{25}$$

Let us consider a continuous (topological) group of coordinate transformation $x \rightarrow x' = x'(x, a)$, and let the field function $u = u(x)$ admits the representation of this group:

$$x \rightarrow x', \quad u(x) \rightarrow u'(x'). \tag{26}$$

The invariance of functional $S[u]$ with respect to (26) means that

$$\int_R d^2x' L[x', u', \partial u' / \partial x'] = \int_R d^2x L[x, u, \partial u / \partial x].$$

The transformation (26) constitutes a group. Therefore, infinitesimal forms of transformations (26) are

$$\Delta x^\nu = X_s^\nu \delta a^s, \quad \Delta u = Y_s \delta a^s \quad (\nu = 0, 1, \quad s = 1, \dots, m), \tag{27}$$

where X_s^ν and Y_s are the generators of the group of transformation correspondingly in the coordinate and the field representations. The index $s = 1, \dots, m$ is defined by the representation of the group.

For simple examples of transformation of the coordinate, time or scalar field, we have

$$\Delta x^\nu = X^\nu \delta a \quad (\nu = 0, 1), \tag{28}$$

$$\Delta u = Y \delta a. \tag{29}$$

Noether’s theorem states that every continuous transformation of coordinate (28) and field function (29), which ensures that the variation of the action is zero admits a conservation law in the form of a continuity equation [27,28]. Substitution of (28), (29), and (15) into (25) gives

$$\delta S[u, h] = \int_R d^2x \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu u)} Y - \theta_\nu^\mu X^\nu \right) \delta a. \tag{30}$$

In view of the fact that the variation of the parameter, δa , is arbitrary, from (30) we get the conservation law

$$\partial_\mu J^\mu = 0, \tag{31}$$

where

$$J^\mu = \frac{\delta L}{\delta(\partial_\mu u)} Y - \theta_\nu^\mu X^\nu \tag{32}$$

is the Noether’s currents and θ_ν^μ is defined in (24). Eq. (31) means that there exists conservation law. Non-triviality of Eqs. (31) and (32) is that L and θ_ν^μ have non-local interaction and memory.

3. Application of the Noether’s theorem for long-range interaction and long-term memory

3.1. Lagrangian functional and energy–momentum tensor

For $\mu = 0, 1$, $x^0 = t$, and $x^1 = r$, the Euler–Lagrange functional Eq. (15) is

$$\frac{\delta L[x, u, \partial_t u, \partial_r u]}{\delta u(x)} - \frac{\partial}{\partial t} \frac{\delta L[x, u, \partial_t u, \partial_r u]}{\delta(\partial_t u(x))} - \frac{\partial}{\partial r} \frac{\delta L[x, u, \partial_t u]}{\delta(\partial_r u(x))} = 0. \tag{33}$$

The time and space variables in action can be separated to consider the field with power-law memory and long-range interaction. Let $K_0(x, y)$ and $K_1(x, y)$ have the form

$$K_0(x, y) = \delta(r - r') \mathcal{K}_0(t, t'), \tag{34}$$

$$K_1(x, y) = \delta(t - t') \mathcal{K}_1(r, r'), \tag{35}$$

and

$$U(u(x), u(y)) = U(u(x))\delta(x - y), \tag{36}$$

where $x = (t, r)$, and $y = (t', r')$. Then the Lagrangian functional is

$$L[x, u, \partial_t u, \partial_r u] = \frac{1}{2} \int dt' \mathcal{K}_0(t, t') \partial_t u \partial_{t'} u - \frac{1}{2} \int dr' \mathcal{K}_1(r, r') \partial_r u \partial_{r'} u - U(u). \tag{37}$$

To present the long-term memory and long-range interaction, consider the kernels $\mathcal{K}_0(t, t')$ and $\mathcal{K}_1(r, r')$ in the power-law forms

$$\mathcal{K}_1(r, r') = \mathcal{K}_1(|r - r'|) = \frac{-g}{\cos(\pi\alpha/2)\Gamma(2 - \alpha)} \frac{1}{|r - r'|^{\alpha-1}} \quad (1 < \alpha < 2), \tag{38}$$

and

$$\mathcal{K}_0(t, t') = \begin{cases} \mathcal{M}(t - t'), & 0 < t' < t, \\ 0, & t' > t, \quad t' < 0, \end{cases} \tag{39}$$

with

$$\mathcal{M}(t - t') = \frac{1}{\Gamma(1 - \beta)} \frac{1}{(t - t')^\beta} \quad (0 < \beta < 1). \tag{40}$$

Then the variational derivatives of Lagrangian functional are

$$\frac{\delta L}{\delta u} = - \frac{\partial U(u(t, r))}{\partial u(t, r)}, \tag{41}$$

$$\frac{\delta L}{\delta(\partial_t u)} = {}_0^C D_t^\beta u(t, r) \quad (0 < \beta < 1), \tag{42}$$

$$\frac{\delta L}{\delta(\partial_r u)} = -g \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t, r) \quad (1 < \alpha < 2), \tag{43}$$

where ${}_0^C D_t^\beta$ is Caputo derivative [14,12] defined by

$${}_0^C D_t^\beta u(t, r) = \frac{1}{\Gamma(n - \beta)} \int_0^t \frac{d\tau}{(t - \tau)^{\beta-n+1}} \frac{\partial^n u(\tau, r)}{\partial \tau^n} \quad (n - 1 < \beta < n).$$

Substitution of (41)–(43) into (33) gives the fractional field Euler–Lagrange equation

$${}_0^C D_t^{\beta+1} u(t, r) - g \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0 \quad (1 < \alpha < 2, 0 < \beta < 1). \tag{44}$$

For $\alpha = 2$ and $\beta = 1$, Eq. (44) is the usual field equation

$$\partial_t^2 u(t, r) - g \partial_r^2 u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0. \tag{45}$$

For the potential $U(u(t, r)) = -\cos u(t, r)$, Eq. (45) gives the sine-Gordon equation,

$$\partial_t^2 u(t, r) - \partial_r^2 u(t, r) + \sin u(t, r) = 0, \tag{46}$$

and Eq. (44) is a spatio-temporal fractional sine-Gordon equation

$$\partial_t^\beta u - \partial_{|r|}^\alpha u + \sin u = 0, \tag{47}$$

where we used for abbreviation simplified notation for fractional derivatives. For the case $\beta = 2$, the equation was obtained in [20].

The energy–momentum tensor θ_v^μ , can be presented in the form

$$\theta_v^\mu = \theta \delta_{\mu\nu} + \tau_v^\mu, \tag{48}$$

where the first term is the diagonal part

$$\theta = \theta_0^0 + \theta_1^1 = \frac{\delta L}{\delta(\partial_t u)} (\partial_t u) + \frac{\delta L}{\delta(\partial_r u)} (\partial_r u) - 2L = U(u(t, r)). \tag{49}$$

This represents a pressure-like quantity. The second term in the right hand side of (48) is a non-diagonal of θ_v^μ :

$$\tau_v^\mu = \frac{\delta L}{\delta(\partial_\mu u)} (\partial_v u) - \frac{\delta L}{\delta(\partial_\kappa u)} (\partial_\kappa u) \delta_v^\mu, \tag{50}$$

where

$$\tau_0^0 = \tau_1^1 = 0, \tag{51}$$

$$\tau_1^0 = \frac{\delta L}{\delta(\partial_t u)} (\partial_t u) = [{}_0^C D_t^\beta u(t, r)] \partial_t u(t, r), \tag{52}$$

$$\tau_0^1 = \frac{\delta L}{\delta(\partial_r u)} (\partial_r u) = -g \left[\frac{\partial^{x-1}}{\partial |r|^{x-1}} u(t, r) \right] \partial_r u(t, r). \tag{53}$$

For $r \in \mathbb{R}^3$ the spatial components of τ_v^μ , represent shear stress tensor. The value θ represents a pressure-like quantity, normal stress.

3.2. Homogeneity in time

The homogeneity in time means invariance of action with respect to the transformation

$$t \rightarrow t' = t + a, \quad r \rightarrow r, \quad u \rightarrow u. \tag{54}$$

Then the infinitesimal transformations are

$$\Delta x^\mu = \delta_0^\mu \delta a, \quad \Delta u = 0 \quad (\mu = 0, 1) \tag{55}$$

with the generators

$$X^\mu = \delta_0^\mu, \quad Y = 0 \quad (\mu = 0, 1). \tag{56}$$

The Noether's current has two components

$$J^\mu = -\theta_0^\mu = -\left(\frac{\delta L}{\delta(\partial_\mu u)} \partial_t u - \delta_0^\mu L \right) \quad (\mu = 0, 1). \tag{57}$$

Using $x^0 = t$, and $x^1 = r$, we get the continuity equation

$$\partial_r J^0 + \partial_r J^1 = 0, \tag{58}$$

where

$$\begin{aligned} J^0 &= -\left(\frac{\delta L}{\delta(\partial_t u)} (\partial_t u) - L \right) = -(\partial_t u(t, r) {}_0^C D_t^\beta u(t, r) - L) \\ &= -\left(\frac{1}{2} \partial_t u(t, r) {}_0^C D_t^\beta u(t, r) + \frac{1}{2} g \partial_t u(t, r) \frac{\partial^{x-1}}{\partial |r|^{x-1}} u(t, r) + U \right), \end{aligned} \tag{59}$$

and

$$J^1 = -\frac{\delta L}{\delta(\partial_r u)} (\partial_r u) = g \partial_t u(t, r) \frac{\partial^{x-1}}{\partial |r|^{x-1}} u(t, r). \tag{60}$$

As a result, the continuity equation is

$$\partial_t \left(\frac{1}{2} \partial_t u {}_0^C D_t^\beta u + g \frac{1}{2} \partial_t u \frac{\partial^{x-1}}{\partial |r|^{x-1}} u + U \right) - \partial_r \left(g \partial_t u \frac{\partial^{x-1}}{\partial |r|^{x-1}} u \right) = 0, \tag{61}$$

where $u = u(t, r)$. Here

$$\mathcal{H} = \frac{1}{2} \partial_t u_0^C D_t^\beta u + \frac{1}{2} g \partial_r u \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u + U \tag{62}$$

is a fractional generalization of density of energy (density of Hamiltonian), and

$$\mathcal{P} = -g \partial_t u \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u \tag{63}$$

is a fractional generalization of density of momentum. For the case $\alpha = 2$ and $\beta = 1$, we obtain the well-known relations [28]:

$$\mathcal{H} = \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} g (\partial_r u)^2 + U, \quad \mathcal{P} = -g \partial_t u \partial_r u. \tag{64}$$

The continuity Eq. (61) can be presented in an usual form of the conservation of energy

$$\partial_t \mathcal{H} + \partial_r \mathcal{P} = 0 \tag{65}$$

with the fractional generalizations of the energy and momentum given by (62) and (63).

3.3. Homogeneity of space

The homogeneity of space means invariance of action with respect to transformations

$$r \rightarrow r' = r + a, \quad t \rightarrow t, \quad u \rightarrow u. \tag{66}$$

The corresponding infinitesimal transformations are presented by

$$\Delta x^\mu = \delta_1^\mu \delta a, \quad \Delta u = 0 \tag{67}$$

with the generators

$$X^\mu = \delta_1^\mu, \quad Y = 0. \tag{68}$$

Then the Noether's current has the following two components

$$J^\mu = -\theta_1^\mu = -\left(\frac{\delta L}{\delta (\partial_\mu u)} (\partial_\mu u) - \delta_1^\mu L \right) \quad (\mu = 0, 1). \tag{69}$$

Using $x^0 = t$, and $x^1 = r$, we get the continuity equation that corresponds to the homogeneity of one-dimensional space

$$\partial_r J^0 + \partial_r J^1 = 0, \tag{70}$$

where

$$J^0 = -\frac{\delta L}{\delta (\partial_t u)} (\partial_t u) = -\partial_r u(t, r)_0^C D_t^\beta u(t, r), \tag{71}$$

and

$$\begin{aligned} J^1 &= -\frac{\delta L}{\delta (\partial_r u)} (\partial_r u) + L = g \partial_r u(t, r) \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t, r) + L \\ &= \frac{1}{2} \partial_t u(t, r)_0^C D_t^\beta u(t, r) + \frac{1}{2} g \partial_r u(t, r) \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t, r) - U. \end{aligned} \tag{72}$$

As a result, the continuity Eq. (70) is

$$-\partial_t (\partial_r u_0^C D_t^\beta u) + \partial_r \left(\frac{1}{2} \partial_t u_0^C D_t^\beta u + \frac{1}{2} g \partial_r u \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u - U \right) = 0 \tag{73}$$

that can be interpreted as the momentum conservation law in the case of fractional dynamics.

For $\beta = 1$ and $\alpha = 2$, Eq. (73) gives

$$-\partial_t(\partial_r u \partial_t u) + \partial_r \left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}g(\partial_r u)^2 - U \right) = 0. \tag{74}$$

Note that, in general, for integer and fractional values of α and β , $J_1^0 \neq J_0^1$, and the energy–momentum tensor θ_v^μ is not a symmetric with respect to μ and v .

3.4. Field translation invariance

For the case $U(u) = 0$, the action (18) is invariant with respect to transformations

$$x^\mu \rightarrow x'^\mu = x^\mu, \quad u \rightarrow u + a. \tag{75}$$

The infinitesimal transformations are presented as

$$\Delta x^\mu = 0, \quad \Delta u = \delta a \tag{76}$$

with the generators

$$X^\mu = 0, \quad Y = 1. \tag{77}$$

The Noether’s current

$$J^\mu = \frac{\delta L}{\delta(\partial_\mu u)} \quad (\mu = 1, 2) \tag{78}$$

has components

$$J^0 = \frac{\delta L}{\delta(\partial_t u)} = {}^C_0 D_t^\beta u(t, r) \quad (0 < \beta < 1), \tag{79}$$

$$J^1 = \frac{\delta L}{\delta(\partial_r u)} = -g \frac{\partial^{x-1}}{\partial |r|^{x-1}} u(t, r) \quad (1 < \alpha < 2). \tag{80}$$

As a result, the continuity equation

$$\partial_\mu J^\mu = \partial_t J^0 + \partial_r J^1 = 0$$

has the form

$$\partial_{i_0} {}^C D_t^\beta u(t, r) - g \partial_r \frac{\partial^{x-1}}{\partial |r|^{x-1}} u(t, r) = 0 \quad (1 < \alpha < 2, \quad 0 < \beta < 1). \tag{81}$$

For $\alpha = 2$ and $\beta = 1$, we get

$$\partial_t^2 u(t, r) - g \partial_r^2 u(t, r) = 0, \tag{82}$$

which is the usual field Eq. (45) for $U(u) = 0$.

Note that [14]

$$\partial_{i_0} {}^C D_t^\beta u(t, r) = {}^C_0 D_t^{1+\beta} u(t, r) + \frac{t^{-\beta}}{\Gamma(1-\beta)} \partial_t u(0, r). \tag{83}$$

To have the relation $\partial_{i_0} {}^C D_t^\beta u(t, r) = {}^C_0 D_t^{1+\beta} u(t, r)$, the initial conditions $\partial_t u(0, r) = 0$ should be applied. In general, Eq. (81) cannot be presented as

$${}^C_0 D_t^{1+\beta} u(t, r) - g \frac{\partial^x}{\partial |r|^x} u(t, r) = 0 \quad (1 < \alpha < 2, \quad 0 < \beta < 1), \tag{84}$$

which is the fractional field Eq. (44) for $U(u) = 0$.

This shows that the conservation law (81), in general, does not coincide with the field equation as it is happened for integer derivatives, unless, we use special boundary and initial conditions.

4. Chain with long-range interaction

It was shown in [20,21] how the long-range interaction between different oscillators can be described by the fractional differential equations in the continuous medium limit. In this section, the Noether’s theorem will be applied to such kind of systems.

4.1. Equation of motion and Noether’s currents

Let us define the action as

$$S[u_n] = \int_{-\infty}^{+\infty} dt \left(\sum_{n=-\infty}^{+\infty} \left[\frac{1}{2} \dot{u}_n^2(t) - V(u_n(t)) \right] - \sum_{\substack{n,m=-\infty \\ m \neq n}}^{+\infty} U(u_n(t), u_m(t)) \right), \tag{85}$$

where u_n is displacement of the n th oscillator from the equilibrium,

$$U(u_n(t)) = \frac{1}{4} g_0 J_\alpha (|n - m|) (u_n(t) - u_m(t))^2, \tag{86}$$

and

$$J_\alpha (|n - m|) = \frac{1}{|n - m|^{\alpha+1}} \quad (\alpha > 0). \tag{87}$$

The Lagrangian of the chain is

$$\mathcal{L} = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2} \dot{u}_n^2(t) - V(u_n(t)) \right] - \frac{1}{4} g_0 \sum_{\substack{n,m=-\infty \\ m \neq n}}^{+\infty} J_\alpha (|n - m|) (u_n(t) - u_m(t))^2. \tag{88}$$

The equation of motion

$$\frac{\partial \mathcal{L}}{\partial u_n(t)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} = 0 \tag{89}$$

for Lagrangian (88) has the form

$$\frac{d^2}{dt^2} u_n(t) + \frac{\partial V(u_n)}{\partial u_n(t)} + g_0 \sum_{\substack{n,m=-\infty \\ m \neq n}}^{+\infty} J_\alpha (|n - m|) [u_m(t) - u_n(t)] = 0. \tag{90}$$

A continuous limit of Eq. (90) can be defined by a transform operation from $u_n(t)$ to $u(x, t)$ [20,21]. First, define $u_n(t)$ as Fourier coefficients of some function $\hat{u}(k, t)$, $k \in [-K/2, K/2]$, i.e.

$$\hat{u}(t, k) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-ikx_n} = \mathcal{F}_\Delta \{u_n(t)\}, \tag{91}$$

where $x_n = n\Delta x$, and $\Delta x = 2\pi/K$ is a distance between nearest particles in the chain, and

$$u_n(t) = \frac{1}{K} \int_{-K/2}^{+K/2} dk \hat{u}(t, k) e^{ikx_n} = \mathcal{F}_\Delta^{-1} \{ \hat{u}(t, k) \}. \tag{92}$$

Secondly, in the limit $\Delta x \rightarrow 0$ ($K \rightarrow \infty$) replace $u_n(t) = (2\pi/K)u(x_n, t) \rightarrow u(x, t)dx$, and $x_n = n\Delta x = 2\pi n/K \rightarrow x$. In this limit, Eqs. (91), (92) are transformed into the integrals

$$\tilde{u}(t, k) = \int_{-\infty}^{+\infty} dx e^{-ikx} u(t, x) = \mathcal{F}\{u(t, x)\} = \lim_{\Delta x \rightarrow 0} \mathcal{F}_{\Delta}\{u_n(t)\}, \tag{93}$$

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \tilde{u}(t, k) = \mathcal{F}^{-1}\{\tilde{u}(t, k)\} = \lim_{\Delta x \rightarrow 0} \mathcal{F}_{\Delta}^{-1}\{\hat{u}(t, k)\}. \tag{94}$$

Applying (91) to (90) and performing the limit (93), we obtain

$$\frac{\partial^2 u(t, x)}{\partial t^2} + g_x \frac{\partial^\alpha u(t, x)}{\partial |x|^\alpha} + \frac{\partial V(u)}{\partial u(t, x)} = 0 \quad (0 < \beta < 2, \quad 1 < \alpha < 2), \tag{95}$$

where

$$g_x = 2g_0(\Delta x)^\alpha \Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \tag{96}$$

is the renormalized constant. Eq. (95) were considered in [20,21].

Consider a continuous transformation of time and field

$$t \rightarrow t', \quad u_n(t) \rightarrow u'_n(t'), \tag{97}$$

which form a continuous group with generators X and Y_n such that the infinitesimal transformations are

$$\Delta t = X\delta a, \quad \Delta u_n = Y_n\delta a. \tag{98}$$

The corresponding Noether’s current is

$$J = \sum_{n=-\infty}^{+\infty} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} Y_n - \theta X, \tag{99}$$

where

$$\theta = \sum_{n=-\infty}^{+\infty} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} \dot{u}_n(t) - \mathcal{L}$$

is the energy. Then

$$J = \sum_{n=-\infty}^{+\infty} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} [Y_n - X\dot{u}_n(t)] + \mathcal{L}X. \tag{100}$$

The equation of conservation law is

$$\frac{d}{dt} J = 0. \tag{101}$$

In the next subsections, we consider examples of the conservation laws.

4.2. Homogeneity of time

The homogeneity of time means the invariance of action with respect to the transformation (compare to (54))

$$t \rightarrow t + a, \quad u_n \rightarrow u_n. \tag{102}$$

Its infinitesimal form is

$$\Delta t = \delta a, \quad \Delta u_n = 0 \tag{103}$$

with generators

$$X = 1, \quad Y_n = 0. \tag{104}$$

Then the Noether’s current is

$$\begin{aligned}
 J_t = -\theta &= - \sum_{n=-\infty}^{+\infty} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} \dot{u}_n(t) + \mathcal{L} = - \sum_{n=-\infty}^{+\infty} \dot{u}_n^2(t) + \mathcal{L} \\
 &= - \sum_{n=-\infty}^{+\infty} \frac{1}{2} \dot{u}_n^2(t) - \sum_{n=-\infty}^{+\infty} V(u_n(t)) - \frac{1}{4} g_0 \sum_{\substack{n,m=-\infty \\ m \neq n}}^{+\infty} J_\alpha(|n-m|)(u_n(t) - u_m(t))^2 \equiv -H,
 \end{aligned}$$

where H is the Hamiltonian. The conservation law $dH/dt = 0$ is in the continuous limit gives the equation

$$\frac{d}{dt} \int_{-\infty}^{+\infty} dr \left[\frac{1}{2} [\partial_t u(t,r)]^2 + V(u(t,r)) + \frac{1}{2} g_\alpha u(t,r) \frac{\partial^\alpha}{\partial |r|^\alpha} u(t,r) \right] = \frac{d}{dt} \int_{-\infty}^{+\infty} dr \mathcal{H} = 0, \tag{105}$$

where $1 < \alpha < 2$, and the density \mathcal{H} of Hamiltonian (62) is introduced.

One can compare this equation with Eq. (65) for $\beta = 1$. Both results coincide if the boundary conditions

$$\lim_{r \rightarrow \pm\infty} \mathcal{P} = 0 \tag{106}$$

is applied.

4.3. Translation invariance

For $V(u_n) = 0$, the action (85) is invariant with respect to the transformations

$$t \rightarrow t, \quad u_n \rightarrow u_n + a, \tag{107}$$

or in the infinitesimal form

$$\Delta t = 0, \quad \Delta u_n = \delta a, \tag{108}$$

with the generators

$$X = 0, \quad Y_n = 1. \tag{109}$$

Then the Noether’s current is

$$J_r = \sum_{n=-\infty}^{+\infty} \frac{\partial \mathcal{L}}{\partial \dot{u}_n(t)} = \sum_{n=-\infty}^{+\infty} \dot{u}_n(t) \equiv P, \tag{110}$$

where P is the total momentum. The conservation law is $dP/dt = 0$.

The continuous limit of this conservation law gives the equation

$$\frac{d}{dt} \int_{-\infty}^{+\infty} dr \partial_t u(t,r) = 0. \tag{111}$$

Let us compare this equation with Eq. (81) that is derived for scalar field $u(t,r)$. Integration of (81) with respect to coordinate r gives

$$\partial_t \int_{-\infty}^{+\infty} dr {}_0^C D_t^\beta u(t,r) - g_\alpha \int_{-\infty}^{+\infty} dr \partial_r \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t,r) = 0 \quad (1 < \alpha < 2, 0 < \beta < 1). \tag{112}$$

Then

$$\partial_t \int_{-\infty}^{+\infty} dr {}_0^C D_t^\beta u(t,r) - g_\alpha \left(\frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t,r) \right)_{-\infty}^{+\infty} = 0. \tag{113}$$

As a result, Eq. (81) coincides with (111), for $\beta = 1$, if we use the boundary conditions

$$\lim_{r \rightarrow \pm\infty} \frac{\partial^{\alpha-1}}{\partial |r|^{\alpha-1}} u(t,r) = 0. \tag{114}$$

5. Fractional Hamilton’s equations

5.1. De Donder–Weyl Hamiltonian

The idea of variation of fields based on a manifestly covariant version of the Hamiltonian formalism in field theory known in the calculus of variation of multiple integrals [36,37] has been proposed by Born and Weyl [38]. The mathematical study of geometrical structures underlying the related aspects of the calculus of variations and classical field theory has been undertaken recently by many authors (see for example [37,39,40]).

The De Donder–Weyl Hamiltonian form of the field equations [36] are

$$\partial_\mu u(x) = \frac{\partial \mathcal{H}}{\partial \pi^\mu}, \quad \partial_\mu \pi^\mu(x) = -\frac{\partial \mathcal{H}}{\partial u}, \tag{115}$$

where

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu u)} \tag{116}$$

is called the multi-momenta, and

$$\mathcal{H}(u, \pi^\mu) = (\partial_\mu u)\pi^\mu - \mathcal{L} \tag{117}$$

is called the De Donder–Weyl Hamiltonian density function, $\mathcal{L} = \mathcal{L}(u, \partial_\mu u)$ is Lagrangian density. These equations are known to be equivalent to the Euler–Lagrange field equations if \mathcal{L} is regular in the sense that

$$\det \left[\frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu u) \partial (\partial_\mu u)} \right] \neq 0. \tag{118}$$

5.2. Hamilton’s equations of integer order

Let us consider Hamiltonian systems in the extended phase space of coordinates (x^μ, u, π^μ) . The evolution of fields is defined by stationary states of the action functional

$$S[u, \pi] = \int [\pi^\mu \partial_\mu u - \mathcal{H}(u, \pi)] d^2x, \tag{119}$$

where the Hamiltonian density \mathcal{H} is defined by (117), both u and π are assumed to be independent functions of $x = (t, r)$. In classical field theory, the evolution of the field $u(x)$ is derived by finding the condition for which the action integral (119) is stationary (a minimum or a saddle point). The action functional (119) can be rewritten as

$$S[u, \pi] = \int \omega_\mu, \tag{120}$$

where

$$\omega_\mu = \pi^\mu du - \mathcal{H}(u, \pi) dx^\mu. \tag{121}$$

is the Poincare–Cartan 1-form or the action 1-form.

The stationary action condition $\delta S = 0$ leads to

$$d\omega_\mu = 0. \tag{122}$$

Here the exterior derivative is

$$d = (dx^\nu)D_{x^\nu} + (du)D_u + (d\pi^\nu)D_{\pi^\nu}, \tag{123}$$

where we put new notation for the derivatives

$$D_{x^\nu} = \frac{\partial}{\partial x^\nu}, \quad D_u = \frac{\partial}{\partial u}, \quad D_{\pi^\nu} = \frac{\partial}{\partial \pi^\nu}. \tag{124}$$

The De Donder–Weyl Hamiltonian form of field equations in (122) can be obtained from the condition that the derivative is exterior one. This condition is equivalent to the stationary action principle $\delta S[u, \pi] = 0$. Then Eq. (121) gives

$$\begin{aligned} d\omega_\mu &= d(\pi^\mu du) - d(\mathcal{H} dx^\mu) \\ &= (D_{x^\nu} \pi^\mu) dx^\nu \wedge du + (D_u \pi^\mu) du \wedge du + (D_{\pi^\nu} \pi^\mu) d\pi^\nu \wedge du - (D_{x^\nu} \mathcal{H}) dx^\nu \wedge dx^\mu - (D_u \mathcal{H}) du \wedge dx^\mu \\ &\quad - (D_{\pi^\nu} \mathcal{H}) d\pi^\nu \wedge dx^\mu. \end{aligned} \tag{125}$$

Using $d\pi \wedge dx = -dx \wedge d\pi$, and $(D_{x^\nu} \mathcal{H}), D_u \pi = 0$, we get

$$d\omega = (D_x \pi + D_u \mathcal{H}) dx \wedge du - (D_\pi \pi du - D_\pi \mathcal{H} dx) \wedge d\pi, \tag{126}$$

where the matrix form of the equation is used. The relation $D_\pi \pi = 1$ gives

$$d\omega = (D_x \pi + D_u \mathcal{H}) dx \wedge du - (Du - D_\pi \mathcal{H} dx) \wedge d\pi. \tag{127}$$

From (122), we have

$$du - D_\pi \mathcal{H} dx = 0, \quad D_x \pi = -D_u \mathcal{H}. \tag{128}$$

As the result, we obtain

$$\frac{\partial u}{\partial x^\mu} = D_{\pi^\mu} \mathcal{H}, \quad \frac{\partial \pi^\mu}{\partial x^\mu} = -D_u \mathcal{H}, \tag{129}$$

which are the well-known De Donder–Weyl Hamilton’s equations (115).

5.3. Fractional Hamilton’s equations

The fractional generalization of the form (121) can be defined [34,32] by

$$\omega(\alpha) = \pi D_s^\alpha u - \mathcal{H}(u, \pi) d_s^\alpha x. \tag{130}$$

It will be called the fractional Poincare–Cartan 1-form or simply the fractional action 1-form. We can consider the fractional exterior derivative of the form (130), and use

$$d^\alpha \omega(\alpha) = 0 \tag{131}$$

to obtain the fractional field equations. Here the fractional exterior derivative is

$$d^\alpha = d_s^\alpha x^\nu D_{x^\nu}^\alpha + d_s^\alpha u D_u^\alpha + d_s^\alpha \pi^\nu D_{\pi^\nu}^\alpha, \quad d_s^\alpha x = \Gamma^{-1}(\alpha + 1) d^\alpha x^\alpha, \tag{132}$$

where $D_{x^\nu}^\alpha, D_u^\alpha, D_{\pi^\nu}^\alpha$ can be fractional derivatives of different types [14]. For example, for $x^\mu \in \mathbb{R}^2$, such that $x = (t, r)$, we use

$$D_x^\alpha = ({}^C D_t^{\alpha_0}, D_r^{\alpha_1}),$$

where ${}^C D_t^{\alpha_0}$ is Caputo fractional derivative, and $D_r^{\alpha_1}$ is the Riesz derivative. Fractional differential forms have been suggested in [34] and it is used to describe dynamical systems [32,35].

Then, by some transformations (see Appendix), one can obtain

$$d^\alpha \omega(\alpha) = [D_x^\alpha \pi + D_u^\alpha \mathcal{H}] d_s^\alpha x \wedge d_s^\alpha u - \left[\frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)} d_s^\alpha u - D_\pi^\alpha \mathcal{H} d_s^\alpha x \right] \wedge d_s^\alpha \pi. \tag{133}$$

Using (133) and (131), we get

$$\frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)} d_s^\alpha u - D_\pi^\alpha \mathcal{H} d_s^\alpha x = 0, \quad D_x^\alpha \pi = -D_u^\alpha \mathcal{H}. \tag{134}$$

For the case $u = u(x)$,

$$d_s^\alpha u = d_s^\alpha x D_x^\alpha u. \tag{135}$$

As the result, we obtain

$$D_{x^\mu}^\alpha u = (\pi^\mu)^{\alpha-1} \Gamma(2-\alpha) D_{\pi^\mu}^\alpha \mathcal{H}, \quad D_{x^\mu}^\alpha \pi^\mu = -D_u^\alpha \mathcal{H}. \tag{136}$$

These equations are the fractional generalization of De Donder–Weyl Hamilton’s equations (129).

As an example, we can consider

$$\mathcal{H}(u, \pi) = \frac{2-\alpha}{2} (\pi^0)^2 - \frac{2-\alpha}{2} (\pi^1)^2 + U(u). \tag{137}$$

Using

$$D_{\pi^\nu}^\alpha (\pi^\mu)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (\pi^\mu)^{\beta-\alpha} \delta_\nu^\mu, \tag{138}$$

and $\Gamma(z+1) = z\Gamma(z)$, Eq. (136) gives

$$D_t^{\alpha_0} u = \pi^0, \quad D_r^{\alpha_1} u = -\pi^1, \tag{139}$$

$$D_t^{\alpha_0} \pi_0 + D_r^{\alpha_1} \pi_1 = -D_u^{\alpha_u} U(u). \tag{140}$$

After substitution of (139) into (140), we obtain

$$(D_t^{\alpha_0})^2 u(t, r) - (D_r^{\alpha_1})^2 u(t, r) + D_u^{\alpha_u} U(u) = 0. \tag{141}$$

For $\alpha_0 = \alpha_1 = \alpha_u = 1$, Eq. (141) is the usual wave equation

$$\partial_t^2 u(t, r) - \partial_r^2 u(t, r) + \frac{\partial U(u)}{\partial u(t, r)} = 0.$$

For $U(u) = 0$, Eq. (141) has the form

$$(D_t^{\alpha_0})^2 u(t, r) - (D_r^{\alpha_1})^2 u(t, r) = 0. \tag{142}$$

The solution $u(t, r)$ of this equation is a linear combination of the solutions $u_1(t, r)$ and $u_2(t, r)$ of the equations

$$D_t^{\alpha_0} u_1(t, r) - D_r^{\alpha_1} u_1(t, r) = 0, \quad D_t^{\alpha_0} u_2(t, r) + D_r^{\alpha_1} u_2(t, r) = 0. \tag{143}$$

For $\alpha_1 = 1$, there exists a relation between the Dirac solutions and the fractional extension of D’Alembert expression that is considered in [41].

Using the property

$$D_x^\alpha D_x^\alpha u = D_x^{2\alpha} u, \tag{144}$$

and we get the fractional equation

$$D_t^{2\alpha_0} u - D_r^{2\alpha_1} u + D_u^{\alpha_u} U(u) = 0. \tag{145}$$

For a special case,

$$U(u) = m^2 \frac{\Gamma(2)}{\Gamma(2+\alpha_u)} u^{1+\alpha_u}. \tag{146}$$

Eq. (145) gives

$$D_t^{2\alpha_0} u(t, r) - D_r^{2\alpha_1} u(t, r) + m^2 u(t, r) = 0. \tag{147}$$

This is a fractional generalization of Klein–Gordon equation (see also [20,30]). Note that Eq. (147) is not Lorentz invariant equations. To obtain fractional relativistic equations, the fractional power of D’Alembertian should be used [42] (see also Section 28 of [11]). The causality principle [28,43] also should be taken into account. For

$$U(u) = \sin(u(t, r)), \tag{148}$$

Eq. (145) gives

$$D_t^{2\alpha_0} u - D_r^{2\alpha_1} u + \sin(u + \alpha_u \pi/2) = 0. \tag{149}$$

This equation is fractional sine-Gordon equation that was considered in [20] for $\alpha_0 = 1$ and $\alpha_u = 2$.

6. Conclusion

Exploiting a variational principle to obtain fractional dynamics seems to be a fairly powerful tool that permits a universal consideration of situations with fractal time and space. In this paper, we have demonstrate it by deriving different equations of using generalized Noether’s theorem. Such equations are similar to the regular conservation laws, although the presence of fractional time derivatives reflects a dissipation. Similar variation of action can be used to derive Hamiltonian type equations although the situation here is not uniquely defined and some freedom of choosing the differential action 1-form leaves different possibilities.

In fact the introduction of Hamiltonian type Eqs. (136) needs a more transparent discussion. It is known that Hamiltonian description of systems can be constructed even for the case of presence of dissipation if one use fractional derivatives (see for example [29,30] and references therein) or interaction of the system with random forces (environment [44] or stochastic processes [31,45]). Nevertheless, such unusual construction of generalized Hamiltonian type equations doesn’t mean the existence of usual preserving variables. Instead, one arrive to some generalized “conservation laws” in the form

$$\partial_t \rho + \text{div} \mathbf{J} = 0, \tag{150}$$

where ρ and \mathbf{J} are defined through the equations with long-range interaction and long-term memory. Eq. (150) has a meaning of conservation of some flow, but it does not imply a unique interpretation depending on the type of the problem, boundary–initial conditions, performed limits in its derivations, etc. The same can be mentioned about the generalized Hamiltonian type Eq. (136). We consider them more as a possible formal tool since no specific application is introduced.

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Appendix

To prove the proposition (133), we use the rule

$$D_x^\alpha (fg) = \sum_{s=0}^{\infty} \binom{\alpha}{s} (D_x^{\alpha-s} f) \frac{\partial^s g}{\partial x^s},$$

and the relation [14]

$$\frac{\partial^s}{\partial x^s} [d_s^\alpha x] = 0 \quad (s \geq 1)$$

for integer s , where

$$\binom{\alpha}{k} = \frac{(-1)^{k-1} \alpha \Gamma(k - \alpha)}{\Gamma(1 - \alpha) \Gamma(k + 1)}.$$

For example, we have

$$d^\alpha [A_\mu d_s^\alpha x^\mu] = \sum_{s=0}^{\infty} d_s^\alpha x^\nu \wedge \binom{\alpha}{s} (D_{x^\nu}^{\alpha-s} A_\mu) \frac{\partial^s}{\partial (x^\nu)^s} d_s^\alpha x^\mu = d_s^\alpha x^\nu \wedge d_s^\alpha x^\mu \binom{\alpha}{0} (D_{x^\nu}^\alpha A_\mu) = (D_{x^\nu}^\alpha A_\mu) d_s^\alpha x^\nu \wedge d_s^\alpha x^\mu.$$

As a result,

$$\begin{aligned} d^\alpha \omega(\alpha) &= d^\alpha(\pi d_s^\alpha u) - d^\alpha(\mathcal{H} d_s^\alpha x) \\ &= (D_x^\alpha \pi) d_s^\alpha x \wedge d_s^\alpha u + (D_u^\alpha \pi) d_s^\alpha u \wedge d_s^\alpha u + (D_\pi^\alpha \pi) d_s^\alpha \pi \wedge d_s^\alpha u - (D_x^\alpha \mathcal{H}) d_s^\alpha x \wedge d_s^\alpha x - (D_u^\alpha \mathcal{H}) d_s^\alpha u \wedge d_s^\alpha x \\ &\quad - (D_\pi^\alpha \mathcal{H}) d_s^\alpha \pi \wedge d_s^\alpha x. \end{aligned} \tag{151}$$

Here $D_{x_0}^\alpha = {}_0^C D_t^{\alpha_0}$ is Caputo fractional derivative, and $D_{x_1}^\alpha = \delta^\alpha / \partial |r|^\alpha$ is the Riesz fractional derivative [14]. Using

$$d_s^\alpha \pi \wedge d_s^\alpha x = -d_s^\alpha x \wedge d_s^\alpha \pi$$

and $D_x^\alpha \mathcal{H}(u, \pi) = 0$, $D_u^\alpha \pi = 0$ for Riesz and Caputo derivatives, we can rewrite Eq. (151) in the form

$$d^\alpha \omega(\alpha) = [D_x^\alpha \pi + D_u^\alpha \mathcal{H}] d_s^\alpha x \wedge d_s^\alpha u - [(D_\pi^\alpha \pi) d_s^\alpha u - (D_\pi^\alpha \mathcal{H}) d_s^\alpha x] \wedge d_s^\alpha \pi. \tag{152}$$

Substitution of

$$D_\pi^\alpha \pi = \frac{\pi^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{153}$$

into Eq. (152) gives (133).

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